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Dynamical Systems

# Billiard in a triangle and quadratic homogeneous foliations on $\mathbf{C}^2$

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**Abstract**

This Note is partly an announcement. We describe the geodesic foliation of the translation surface associated to a triangular billiard in terms of real and holomorphic foliations defined by quadratic homogeneous vector fields on $\mathbf{C}^2$. Our technique allows, in particular, to state the existence of real foliations by curves on $\mathbf{C}^2$ arising from homogeneous quadratic vector fields and having a dense set of periodic orbits. ***To cite this article: F. Valdez, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***


**Résumé**

**Billard sur un triangle et feuilletages homogènes quadratiques de $\mathbf{C}^2$.** Cette Note est partiellement une annonce. Nous décrivons le feuilletage géodésique de la surface de translation associée au billiard sur un triangle en termes de feuilletages réels et holomorphes définis par des champs de vecteurs quadratiques homogènes de $\mathbf{C}^2$. Nos techniques nous permettent, entre autres, de constater l'existence des feuilletages réels en courbes de $\mathbf{C}^2$ définis par des champs de vecteurs homogènes, quadratiques et pour lesquels les orbites périodiques sont denses. ***Pour citer cet article : F. Valdez, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***


## Version française abrégée

Soit $\mathcal{F}_\lambda$ le feuilletage holomorphe de $\mathbf{C}^2$ formé par les courbes intégrales du champ de vecteurs

$$X_\lambda := z_1\big(\lambda_3 z_2 + \lambda_2(z_2 - z_1)\big)\partial/\partial z_1 + z_2\big(\lambda_3 z_1 + \lambda_1(z_1 - z_2)\big)\partial/\partial z_2, \tag{1}$$

où $\sum \lambda_j = 1$ et $0 < \lambda_j$, $j = 1, 2, 3$. Possons $\lambda = (\lambda_1, \lambda_2, \lambda_3)$. Ce feuilletage est homogène, i.e., invariant par le groupe d'homothéties de $\mathbf{C}^2$, à singularité isolée et cône tangent réduit. Le cône tangent de $\mathcal{F}_\lambda$ est formé par trois droites passant par l'origine. Les feuilles de $\mathcal{F}_\lambda$ différentes du cône tangent sont appelées *feuilles génériques*. Deux feuilles génériques sont difféomorphes *via* une homothétie de $\mathbf{C}^2$.

**Théorème 0.1.** *Si le $\mathbf{Z}$-sous-module $Res(\lambda) := \{(n_1, n_2, n_3) \in \mathbf{Z}^3 \mid \sum_j n_j\lambda_j = 0\}$ de $\mathbf{Z}^3$ est trivial, alors la feuille générique de $\mathcal{F}_\lambda$ est homéomorphe au monstre de Loch Ness, c'est-à-dire, au plan réel auquel on a attaché une infinité dénombrable d'anses.*

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La preuve de ce théorème n'est pas presentée dans cette note. Soit $\mathcal{F}_{\lambda,0}$ le feuilletage réel de $\mathbf{C}^2$ défini par les courbes intégrales du champ de vecteurs $Re(X_\lambda)$. Nous décrivons ce feuilletage en termes du jeu du billard sur le triangle $P \subset \mathbf{R}^2$ d'angles $\{\lambda_j \pi\}_{j=1}^3$. Tout au long de cet article on suppose $P$ privé de ses sommets. Soit $H_\lambda$ le groupe des isométries de $\mathbf{R}^2$ engendré par les réflexions dont les axes sont les droites contenannt les côtés de $P$. Lorsqu'on identifie dans $\bigsqcup_{h \in H_\lambda} hP$ deux copies de $P$ différant par une translation du plan, on obtient une surface de translation $S(P)$ [5]. L'étude du jeu du billard sur $P$ se ramène à l'étude des feuilletages géodesiques de $S(P)$ associés à la métrique plate provenant de $\mathbf{R}^2$. Ces feuilletages sont formés par des géodesiques parallèles à une direction fixée $\theta \in \mathbf{R}/2\pi\mathbf{Z}$.

Toute courbe intégrale du champ $X_\lambda$ est naturellement munie d'une structure de surface de translation : les cartes correspondent au « flot local » associé au champ $X_\lambda$. Nous prouvons

**Théorème 0.2.** *En tant que surface de translation, toute feuille générique de $\mathcal{F}_{a,\lambda}$ est isomorphe à la surface $S(P)$.*

En particulier, pour toute feuille générique $\mathcal{L} \in \mathcal{F}_\lambda$ il existe une direction $\theta \in \mathbf{R}/2\pi\mathbf{Z}$ telle que le feuilletage $\mathcal{F}_{\lambda,0|\mathcal{L}}$ est analytiquement conjugué au feuilletage de la surface de translation $S(P)$ formé par les géodésiques parallèles à la direction $\theta$. Pour tout $\rho \in \mathbf{R}^*$, $\theta \in \mathbf{R}/2\pi\mathbf{Z}$ et $\mathcal{L}$ générique nous désignons par $\rho e^{i\theta}\mathcal{L}$ l'image de l'homothétie $(z_1, z_2) \rightsquigarrow \rho e^{i\theta}(z_1, z_2)$ restreinte à la feuille $\mathcal{L}$. On prouve aussi que, si $\mathcal{F}_{\lambda,0|\mathcal{L}}$ est analytiquement conjugué au feuilletage de la surface de translation $S(P)$ formé par les géodésiques parallèles à la direction $\theta$, alors pour tout $\theta' \in \mathbf{R}/2\pi\mathbf{Z}$ et $\rho \in \mathbf{R}^*$, $\mathcal{F}_{\lambda,0|\rho e^{i\theta'}\mathcal{L}}$ est conjugué au feuilletage de $S(P)$ défini par les géodésiques parallèles à la direction $\theta + \theta'$.

Ce théorème entraîne, entre autres, que les orbites périodiques de $\mathcal{F}_{\lambda,0}$ sont denses dans $\mathbf{C}^2$ lorsque $\lambda \in \mathbf{Q}^3$. Pour ce type de paramètres $\lambda$, la mesure de LEBESGUE de l'ensemble de feuilles $\mathcal{L} \in \mathcal{F}_\lambda$ où $\mathcal{F}_{\lambda,0|\mathcal{L}}$ n'est pas minimal est nulle. En gros, le théorème précédent établit un « dictionnaire » entre les feuilletages du type $\mathcal{F}_{\lambda,0|\mathcal{L}}$, $\mathcal{L} \in \mathcal{F}_\lambda$ générique, et le billard sur un triangle.

Nous associons aux feuilletages $\mathcal{F}_\lambda$ et $\mathcal{F}_{\lambda,0}$ via la projection naturelle de $\mathbf{R}^4 \setminus 0$ sur $\mathbf{RP}(3)$ des feuilletages singuliers $\mathcal{G}_\lambda$ et $\mathcal{G}_{\lambda,0}$ de codimension réelle un et deux respectivement. Si $\lambda$ est *non fortement résonnant*, notion que nous précisons plus loin, le feuilletage $\mathcal{G}_\lambda$ a sa feuille générique difféomorphe à la feuille générique de $\mathcal{F}_\lambda$. Nous obtenons ainsi une version *projective* du résultat précédent (voir Corollaire 3.6). Nous menons une description de $\mathcal{G}_{\lambda,0}$ en termes du billard sur le triangle $P$. Par exemple, le lieu singulier de ce feuilletage réel est formé par trois points représentant, dans un sens que nous préciserons plus loin, l'ensemble de sommets du triangle $P$. Soit $\widetilde{\mathbf{RP}(3)}$ la 3-variété qui résulte d'éclater $\mathbf{RP}(3)$ le long du lieu singulier $\mathcal{G}_\lambda$. Le feuilletage $\mathcal{G}_{\lambda,0}$ définit naturellement un feuilletage réel $\widetilde{\mathcal{G}_{\lambda,0}}$ de cette 3-variété. Nous menons une description de $\widetilde{\mathcal{G}_{\lambda,0}}$ dans un voisinage des diviseurs exceptionnels à l'aide d'une intégrale première pour $\mathcal{G}_{\lambda,0}$ et le billard sur le triangle $P$. Cette description montre de nouvelles informations sur la dynamique du billard qui se « trouvent dans les sommets du triangle ».

## 1. Introduction

Let $\mathcal{F}_\lambda$ be the holomorphic foliation on $\mathbf{C}^2$ defined by the integral curves of the vector field (1). We note that $\{\lambda_j \pi\}_{j=1}^3$ are the angles of a non-degenerated triangle $P$ in the plane $\mathbf{C}$. Through this Note we assume that $P$ has no vertices. We remark that $F_\lambda(z_1, z_2) = z_1^{\lambda_1} z_2^{\lambda_2}(z_2 - z_1)^{\lambda_3}$ is a first integral for $\mathcal{F}_\lambda$. This foliation is *homogeneous*, i.e. invariant under the natural action of the homothetic transformation group $\{T_k(z_1, z_2) := k(z_1, z_2) \mid k \in \mathbf{C}^*\}$. The foliation $\mathcal{F}_\lambda$ presents an isolated singularity at the origin and leaves only three lines invariant through this point. In the complement of the tangent cone $z_1 z_2(z_2 - z_1) = 0$ any two leaves $\mathcal{L}, \mathcal{L}' \in \mathcal{F}_\lambda$ are diffeomorphic, for there always exist $k \in \mathbf{C}^*$ such that $T_{k|\mathcal{L}} : \mathcal{L} \to \mathcal{L}'$ is a diffeomorphism. We conclude that, up to diffeomorphism, $\mathcal{F}_\lambda$ presents three kinds of leaves: a point, $\mathbf{C}^*$ and a *generic leaf*. The following theorem is stated without proof:

**Theorem 1.1.** *If the $\mathbf{Z}$-submodule $Res(\lambda) := \{(n_1, n_2, n_3) \in \mathbf{Z}^3 \mid \sum_j n_j \lambda_j = 0\}$ of $\mathbf{Z}^3$ is equal to zero, then the generic leaf of $\mathcal{F}_\lambda$ is homeomorphic to a plane with a countable set of handles. This topological surface is also known as the* Loch Ness monster.

Let $\mathcal{F}_{\lambda,0}$ be the real foliation on $\mathbf{C}^2$ defined by the real analytic vector field $Re(X_\lambda)$. This foliation has an isolated singularity at the origin and for every non-singular leaf $\mathcal{L} \in \mathcal{F}_\lambda$ the restriction $\mathcal{F}_{\lambda,0|\mathcal{L}}$ is a real foliation without singularities. We describe this foliation in terms of the billiard game inside the triangle $P$.

A frictionless point inside $P$ moves along a *billiard trajectory*, as time runs from $-\infty$ to $\infty$, if it moves with constant velocity in the interior of $P$ and reflects off edges so that speed is unchanged and the angle of incidence is equal to the angle of reflection. Motion ends at the vertices of $P$, for reflection is not well defined. Let $H_\lambda$ be the group of isometries of $\mathbf{R}^2$ generated by those reflections whose axes contain the sides of $P$. The identification in $\bigsqcup_{h\in H_\lambda} hP$ of any two copies of this triangle differing by a translation leads to a translation surface $S(P)$ [5]. The billiard dynamics inside $P$ can be understood by studying the geodesic foliations on $S(P)$ with respect to the flat metric obtained by lifting the natural flat metric of $\mathbf{R}^2$. These real foliations are formed by geodesics parallel to a given fixed direction $\theta \in \mathbf{R}/2\pi\mathbf{Z}$.

Every non-degenerated integral curve of $X_\lambda$ has a natural translation surface structure. The charts are given by the local flow associated to $X_\lambda$. Our main result states:

**Theorem 1.2.** *Every generic leaf of $\mathcal{F}_\lambda$ is isomorphic,* as translation surface, *to the surface $S(P)$.*

In particular, for every generic leaf $\mathcal{L} \in \mathcal{F}_\lambda$ there exists a direction $\theta \in \mathbf{R}/2\pi\mathbf{Z}$ such that the foliation $\mathcal{F}_{\lambda,0|\mathcal{L}}$ is holomorphically conjugated to the geodesic foliation defined by the direction $\theta$ on the translation surface $S(P)$. Let $\rho \in \mathbf{R}^*$ and $\theta \in \mathbf{R}/2\pi\mathbf{Z}$. We denote $\rho e^{i\theta}\mathcal{L}$ the image of the homothetic transformation $T_{\rho e^{i\theta}|\mathcal{L}}$. Our arguments will show as well that, if $\mathcal{F}_{\lambda,0|\mathcal{L}}$ is holomorphically conjugated to the geodesic foliation defined by the direction $\theta$ on the translation surface $S(P)$, then for every $\theta' \in \mathbf{R}/2\pi\mathbf{Z}$ and $\rho \in \mathbf{R}^*$, $\mathcal{F}_{\lambda,0|\rho e^{i\theta'}\mathcal{L}}$ is holomorphically conjugated to geodesic foliation defined by the direction $\theta + \theta'$ on $S(P)$. Our main result establishes the bases for a “dictionary” between homogeneous foliations defined by (1) and the billiard game in a triangle. The rest of this article is devoted to prove this result and to discuss some applications in both domains.

## 2. Proof, main result

Let $\widetilde{\mathbf{CP}(2)}$ be the surface obtained from $\mathbf{CP}(2)$ after blowing-up the point $z_1 = z_2 = 0$ in the affine coordinates $z_3 = 1$. We denote $\Pi$ the natural projection onto the exceptional divisor. $\mathcal{F}_\lambda$ extends to a foliation $\widetilde{\mathcal{F}_\lambda}$ in $\widetilde{\mathbf{CP}(2)}$. The invariant lines through the origin in $\mathcal{F}_\lambda$ determine three points in the exceptional divisor. In local coordinates $z_2 = tz_1$ these points are given by $t = 0, 1, \infty$. Let $E$ be their complement in the exceptional divisor. $E$ is homeomorphic to the sphere with three punctures. The foliation $\widetilde{\mathcal{F}_\lambda}$ presents a *generic leaf* isomorphic, as translation surface, to the generic leaf of $\mathcal{F}_\lambda$. Every generic leaf $\mathcal{L} \in \widetilde{\mathcal{F}_\lambda}$ defines a covering space

$$\Pi_{|\mathcal{L}} : \mathcal{L} \longrightarrow E. \tag{2}$$

The SCHWARZ–CHRISTOFFEL transformation

$$T : t \to \int_0^t \xi^{\lambda_2 - 1}(\xi - 1)^{\lambda_3 - 1}\,\mathrm{d}\xi, \quad t \in E \tag{3}$$

maps conformally $\mathrm{Im}(t) > 0$ onto the interior of $P$ [2]. SCHWARZ's reflection principle implies that the analytic continuation of $T$ with respect to a loop $m \in \pi_1(E, \frac{1}{2})$ is given by $\sigma_m \circ T$, where $\sigma_m$ in an isometry in the group $H_\lambda$ used to construct the surface $S(P)$. Consider the set of generators $\gamma_2(s) := e^{2\pi i s}/2$ and $\gamma_3(s) := \gamma_2(s) + 1$, $s \in \mathbf{R}$. Up to a change of coordinates, one can assume that $T(\frac{1}{2})$ is real and $\lim_{t\to 0} T(t) = 0$. Then, $\sigma_{\gamma_2} \circ T = e^{2\pi i\lambda_2}T$ and $\sigma_{\gamma_3} \circ T = e^{2\pi i\lambda_3}T + T_0$ for a certain $T_0 \neq 0$. Since $F_\lambda(z_1, t) = z_1 t^{\lambda_2}(t-1)^{\lambda_3}$ is a first integral for $\widetilde{\mathcal{F}_\lambda}$ in local coordinates, the action of $\gamma_j([0,1])$ on the fiber $\eta \in \pi^{-1}(\frac{1}{2})$, is given, up to a change of sign in the exponent, by $\eta \to e^{2\pi i\lambda_j}\eta$. Then, every translation in $H_\lambda$ is of the form $\sigma_m \circ T$, where $m$ is a loop acting trivially on the fiber $\Pi^{-1}(\frac{1}{2})$ and vice versa. This implies that the complex structure defined by $\{\sigma_m \circ T \circ \Pi_{|\mathcal{L}} \mid m \in \pi_1(E, \frac{1}{2})\}$ corresponds to the translation surface structure of $S(P)$.

Let $\widetilde{X_\lambda}$ and $\widetilde{\mathrm{Re}(X_\lambda)}$ be the vector fields naturally defined on $\widetilde{\mathbf{CP}(2)}$ by $X_\lambda$ and $\mathrm{Re}(X_\lambda)$. A straight computation shows that, in local coordinates $z_2 = tz_1$, the projection of $\widetilde{X_\lambda}_{|\mathcal{L}}$ to the exceptional divisor $E$ is given by the branches of $kt^{1-\lambda_2}(t-1)^{1-\lambda_3}\partial/\partial t$, where $k \in \mathbf{C}^*$ is a constant given by the first integral $F_\lambda(z_1, z_2)$ and depending on the leaf $\mathcal{L}$. From (3) we conclude that $T \circ \Pi$ *locally* rectifies the real vector field $\widetilde{\mathrm{Re}(X_\lambda)}$. This proves our claim.

Abusing notation, let $\rho e^{i\theta'}\mathcal{L}$ be the image of the homothetic transformation $T_{\rho e^{i\theta'}|\mathcal{L}}$ in $\widetilde{\mathbf{CP}(2)}$. The projection of $\widetilde{X_\lambda}_{|\rho e^{i\theta'}\mathcal{L}}$ to $E$ is given by the branches of $\rho e^{i\theta'}kt^{1-\lambda_2}(t-1)^{1-\lambda_3}\partial/\partial t$. Therefore, if $\mathcal{F}_{\lambda,0|\mathcal{L}}$ is conjugated to the geodesic foliation on $S(P)$ defined by the direction $\theta$, then $\mathcal{F}_{\lambda,0|\rho e^{i\theta'}\mathcal{L}}$ is conjugated to the geodesic foliation defined by $\theta + \theta'$.

## 3. Applications

Every leaf in $\mathcal{F}_{\lambda,0}$ homeomorphic to $\mathbf{R}/2\pi\mathbf{Z}$ is usually called a *periodic orbit*. Let $\mathcal{L} \in \mathcal{F}_\lambda$ be a generic leaf and $M(\mathcal{L}) := \{e^{i\theta}\mathcal{L} \mid \theta \in \mathbf{R}/2\pi\mathbf{Z}\}$. As a consequence of Theorem 1.2, every foliation formed by parallel geodesics on the translation surface $S(P)$ is holomorphically conjugated to the restriction of $\mathcal{F}_{\lambda,0}$ to a leaf of $\mathcal{F}_\lambda$ in $M(\mathcal{L})$. Following this fact we obtain:

**Theorem 3.1.** *If the set of points in the phase space of the billiard in the triangle $P$ defining a closed trajectory form a dense set, then periodic orbits of $\mathcal{F}_{\lambda,0}$ are dense in $\mathbf{C}^2$.*

The complete proof of this theorem is not included in this Note. From

**Theorem 3.2.** *(See [1,4].) Let $P$ be a right or rational triangle. The periodic orbits of the billiard flow in the phase space $P \times \mathbf{R}/2\pi\mathbf{Z}$ are dense.*

we conclude the following:

**Corollary 3.3.** *If $(\lambda_1, \lambda_2, \lambda_3) \in \mathbf{Q}^3$, then periodic orbits of $\mathcal{F}_{\lambda,0}$ are dense in $\mathbf{C}^2$. If $\lambda_j = 1/2$ for some $j = 1, 2, 3$, then periodic orbits of $\mathcal{F}_{\lambda,0}$ are dense in $\mathbf{C}^2$.*

A foliation on a manifold $M$ is called *minimal* if every leaf is dense in $M$. Using

**Theorem 3.4.** *(See [3].) Let $P$ be rational. For all but countably many directions $\theta \in \mathbf{R}/2\pi\mathbf{Z}$, the geodesic foliation on $S(P)$ defined by $\theta$ is minimal.*

we obtain

**Corollary 3.5.** *If $(\lambda_1, \lambda_2, \lambda_3) \in \mathbf{Q}^3$, then the set of leaves $\mathcal{L}$ in $\mathcal{F}_\lambda$ where $\mathcal{F}_{\lambda,0|\mathcal{L}}$ is not minimal has* LEBESGUE *measure zero.*

### *3.1. Dictionary, projective version*

Let $\Pi_{\mathbf{RP}(3)} : \mathbf{C}^2 \setminus 0 \to \mathbf{RP}(3)$ be the natural projection. Then $\{\Pi_{\mathbf{RP}(3)}(\rho\mathcal{L}) \mid \rho \in \mathbf{R}^*, \mathcal{L} \in \mathcal{F}_\lambda \setminus 0\}$ defines a singular foliation $\mathcal{G}_\lambda$ on the projective space $\mathbf{RP}(3)$. This foliation presents a generic leaf and a singular locus formed by three projective lines $\mathbf{RP}(1)$ corresponding to the projection of the three invariant complex lines of $\mathcal{F}_\lambda$ through the origin. The holonomy group of the foliation $\mathcal{F}_\lambda$ is generated by the maps $z \rightsquigarrow e^{2\pi i\lambda_j}z$, $j = 1, 2, 3$. When this group does not contain the involution $z \rightsquigarrow -z$, the generic leaves of $\mathcal{F}_\lambda$ and $\mathcal{G}_\lambda$ are diffeomorphic. The diffeomorphism between generic leaves is given by the restriction of $\Pi_{\mathbf{RP}(3)}$. In this case we say that the parameter $\lambda$ is not *strongly resonant*. From now on we suppose that $\lambda$ in not strongly resonant. In the same way, $\mathcal{F}_{\lambda,0}$ defines a real singular foliation $\mathcal{G}_{\lambda,0}$ on the projective space. The singular locus of this real foliation is formed by three points, each one contained in one of the projective lines forming the singular locus of $\mathcal{G}_\lambda$. Every generic leaf in $\mathcal{G}_\lambda$ inherits a translation surface structure from the corresponding generic leaf of $\mathcal{F}_\lambda$.

**Corollary 3.6.** *Every generic leaf of $\mathcal{G}_\lambda$ is isomorphic,* as translation surface, *to the surface $S(P)$.*

In particular, for every generic leaf $\mathcal{L} \in \mathcal{G}_\lambda$ there exist a direction $\theta \in \mathbf{R}/2\pi\mathbf{Z}$ such that the foliation $\mathcal{G}_{\lambda,0|\mathcal{L}}$ is holomorphically conjugated to the geodesic foliation defined by the direction $\theta$ on the translation surface $S(P)$. For every leaf $\Pi_{\mathbf{RP}(3)}(\mathcal{L}') = \mathcal{L} \in \mathcal{G}_\lambda$, $\mathcal{L}' \in \mathcal{F}_\lambda$ generic, we denote $\rho e^{i\theta}\mathcal{L} := \Pi_{\mathbf{RP}(3)}(\rho e^{i\theta}\mathcal{L}')$. We deduce as well that, if $\mathcal{G}_{\lambda,0|\mathcal{L}}$ is holomorphically conjugated to the geodesic foliation defined by the direction $\theta$ on the translation surface $S(P)$, then for every $\theta' \in \mathbf{R}/2\pi\mathbf{Z}$, $\mathcal{G}_{\lambda,0|e^{i\theta'}\mathcal{L}}$ is holomorphically conjugated to geodesic foliation defined by the direction $\theta + \theta'$ on $S(P)$. This result establishes a 'correspondence' between geodesics in the unitary tangent bundle of the translation surface $S(P)$ and leaves of $\mathcal{G}_{\lambda,0}$ in the complement of the singular locus of $\mathcal{G}_\lambda$. In particular, every closed geodesic in $S(P)$ 'corresponds' to a periodic orbit in $\mathcal{G}_{\lambda,0}$.

**Definition 3.7.** Let $l$ be a non-compact leaf of a foliation in a (real or complex) manifold $M$. A point $p \in M$ is called *an extremity* of $l$ if and only if there exists a parametrization $\gamma : ]0, +\infty[ \to l$ such that $\lim_{s\to+\infty} \gamma(s) = p$. We say that the extremity $p$ is an analytic extremity if and only if the closure of $l$ in a neighborhood of $p$ is analytic.

**Theorem 3.8.** *Every non-compact leaf of $\mathcal{G}_{\lambda,0}$ has either* 0, 1 *or* 2 analytic *extremities in the singular locus. The set of leaves in $\mathcal{G}_{\lambda,0}$ having two extremities is countable and each singularity in $\mathcal{G}_{\lambda,0}$ is an analytic extremity for a countable set of this kind of leaves.*

We sketch the proof of this theorem. Loosely speaking, singularities of the foliation $\mathcal{G}_{\lambda,0}$ play the role of vertices in the triangle $P$. In this way, every leaf in $\mathcal{G}_{\lambda,0}$ having an extremity 'corresponds' to a billiard trajectory that meets a vertex of the triangle $P$. Therefore leaves having two extremities correspond to *generalized diagonals* in the billiard. The set of generalized diagonals is at most countable [3].

Let $\widetilde{\pi} := \widetilde{\mathbf{RP}(3)} \to \mathbf{RP}(3)$ be the blowup on the singular locus of $\mathcal{G}_\lambda$. Each one of the exceptional divisors resulting from $\widetilde{\pi}$ is a torus $\mathbf{RP}(1) \times \mathbf{RP}(1)$. We denote them $\{\mathbb{T}^2_j\}_{j=1}^3$. Let $\widetilde{\mathcal{G}_\lambda}$ and $\widetilde{\mathcal{G}_{\lambda,0}}$ be the foliations on this manifold induced by $\mathcal{G}_\lambda$ et $\mathcal{G}_{\lambda,0}$ respectively. The foliation $\widetilde{\mathcal{G}_\lambda}$ is not singular and presents a generic leaf that is not homeomorphic to the generic leaf of $\mathcal{F}_\lambda$. In the complement of exceptional divisors, the foliation $\widetilde{\mathcal{G}_{\lambda,0}}$ is isomorphic to $\mathcal{G}_{\lambda,0}$. Thence, the study of this real foliation on $\widetilde{\mathbf{RP}(3)}$ is also related to the billiard game on the triangle $P$. Let $\{p_j\}_{j=1}^3$ denote the singularities of $\mathcal{G}_{\lambda,0}$. The singular locus of $\widetilde{\mathcal{G}_{\lambda,0}}$ is given by $\{\widetilde{\pi}^{-1}(p_j)\}_{j=1}^3$. Each connected component of this singular locus is homeomorphic to a circle $\mathbf{R}/2\pi\mathbf{Z}$ contained in an exceptional divisor $\mathbb{T}^2_j$.

**Theorem 3.9.** *The restriction of $\widetilde{\mathcal{G}_\lambda}$ to each torus $\mathbb{T}^2_j$ defines a* linear *foliation.*

Let $z_j = x_j + iy_j$. This result follows from the fact that in homogeneous coordinates $\lambda_1 \arctan y_1/x_1 + \lambda_2 \arctan y_2/x_2 + \lambda_3 \arctan((y_2 - y_1)/(x_2 - x_1))$ is a first integral for the foliation $\mathcal{G}_\lambda$. We conclude that the singular locus of $\widetilde{\mathcal{G}_{\lambda,0}}$ is formed by apparent singularities and therefore $\widetilde{\mathcal{G}_{\lambda,0}}_{|\mathbb{T}^2_j}$ is a line foliation. This line foliation 'governs' the foliation $\widetilde{\mathcal{G}_{\lambda,0}}$ in a neighborhood of each torus $\mathbb{T}^2_j$. Loosely speaking, each leaf of $\widetilde{\mathcal{G}_{\lambda,0}}$ visiting a neighborhood of $\mathbb{T}^2_j$ describes a finite number of 'turns' around the torus before leaving. In the same way, a billiard trajectory in a neighborhood of a vertex (not meeting a vertex) reflects a finite number of times on the edges defining the vertex before leaving. One can think of the foliation $\widetilde{\mathcal{G}_{\lambda,0}}_{|\mathbb{T}^2_j}$ as representing the billiard dynamics in a vertex of the triangle $P$.

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