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Partial Differential Equations

On the perturbation of the electromagnetic energy due to the presence of small inhomogeneities

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Abstract

We consider solutions to the time-harmonic Maxwell problem in $\mathbf{R}^3$. For such solution we propose a rigorous derivation of the asymptotic expansions in the interesting practical situation when a finite number of inhomogeneities of small diameter are embedded in the entire space. Then, we describe the behavior of the electromagnetic energy caused by the presence of these inhomogeneities. ***To cite this article: C. Daveau, A. Khelifi, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Sur la perturbation de l'énergie électromagnétique due à la présence d'inhomogénéités de petits diamètres. Nous considérons des solutions des équations de Maxwell dans $\mathbf{R}^3$ en présence d'un nombre fini d'inhomogénéités de petits diamètres. Pour de telles solutions, nous obtenons des formules asymptotiques rigoureuses. Puis, nous décrivons le comportement de l'énergie électromagnétique. ***Pour citer cet article : C. Daveau, A. Khelifi, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Version française abrégée

L'objectif du travail décrit dans cette Note est de comprendre comment les solutions perturbées du probleme (3) se comportent quand un nombre fini d'inhomogénéités $\{z_j + \alpha B_j\}$ de petits diamètres sont introduites dans l'espace entier $\mathbf{R}^3$. Ceci nous amène, également, à étudier l'évolution de l'énergie électromagnétique correspondante selon cette déformation du milieu de propagation.

On suppose que dans $\mathbf{R}^3$ on a m inhomogénéités $\{z_j + \alpha B_j\}_{j=1}^m$, où α est un petit paramètre, $B_j \subset \mathbf{R}^3$ est un ouvert borné de diamètre borné indépendamment de j et les points $\{z_j\}_{j=1}^m$ vérifient l'hypothèse (1). On suppose que la perméabilité magnétique μ_α et la permittivité électrique ε_α vérifient (2). Nous commençons notre analyse dans la section 2 en obtenant rigoureusement des développements asymptotiques des champs électrique et magnétique, avec des estimations d'erreurs uniformes dans $\mathbf{R}^3$. Ces formules asymptotiques sont construites par la méthode des déve-

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loppements asymptotiques raccordés. Concernant cette méthode, le lecteur peut consulter [7,10]. En domaine borné, on peut voir aussi les travaux [1,3,5]. Le terme d'ordre principal dans des développements asymptotiques analogues (mais pour le cas d'un domaine borné) a été obtenue par Vogelius et Volkov [14] et Ammari et al. [4]. Nos formules asymptotiques utilisent des tenseurs de polarisation associés à des inhomogénéités électromagnétiques qui semblent être des généralisations naturelles des tenseurs qui ont été présentés par Schiffer et Szegö [13] et complètement étudiés par beaucoup d'autres auteurs [2,6,8,11].

1. Problem formulation

We suppose that there is a finite number of electromagnetic inclusions in $\mathbf{R}^3$, each of the form $z_j + \alpha B_j$, where $B_j \subset \mathbf{R}^3$ is a bounded and $\mathcal{C}^\infty$-domain containing the origin. The total collection of inhomogeneities is:

$$\mathcal{B}_\alpha = \bigcup_{j=1}^{m} \{z_j + \alpha B_j\}.$$

The points $z_j \in \mathbf{R}^3$, $j = 1, \dots, m$, which determine the location of the inhomogeneities, are assumed to satisfy the following inequality:

$$|z_j - z_l| \geqslant c_0 > 0, \quad \forall j \neq l. \tag{1}$$

Let μ_0 and ε_0 denote the permeability and the permittivity of the free space, $\omega > 0$ is a given frequency and $k = \omega\sqrt{\varepsilon_0\mu_0} > 0$ is the wave number. We denote by $x = (x_1, x_2, x_3)$ the Cartesian coordinates in $\mathbf{R}^3$. We shall assume that $\mu_0 > 0$ *and* $\varepsilon_0 > 0$ *are positive constants*. Let $\mu_j > 0$ and $\varepsilon_j > 0$ denote the permeability and the real permittivity of the j-th inhomogeneity, $z_j + \alpha B_j$; these are also assumed to be positive constants. Introduce the piecewise-constant magnetic permeability

$$\mu_\alpha(x) := \begin{cases} \mu_0, & x \in \mathbf{R}^3 \setminus \bar{\mathcal{B}}_\alpha, \\ \mu_j, & x \in z_j + \alpha B_j,\ j = 1, \dots, m. \end{cases} \tag{2}$$

The piecewise-constant electric permittivity $\varepsilon_\alpha(x)$ is defined analogously. If we allow the degenerate case $\alpha = 0$, then the function $\mu_0(x)$ (resp. $\varepsilon_0(x)$) takes the constant value of μ_0 (resp. ε_0). We now suppose that the collection of inclusions $\mathcal{B}_\alpha$ is contained in an open subset Ω of $\mathbf{R}^3$ and that the source current density $\mathbf{J}_s$ is supported in $\mathbf{R}^3 \setminus \bar{\Omega}$.

Let $(\mathbf{E}_\alpha, \mathbf{H}_\alpha) \in \mathbf{R}^3 \times \mathbf{R}^3$ denote the time-harmonic electromagnetic fields, with pulsation ω, in the presence of the electromagnetic inclusions $\mathcal{B}_\alpha$. These time-harmonic fields are solutions of the Maxwell's equations, see [9,12]:

$$\begin{cases} \nabla \times \mathbf{E}_\alpha = i\omega\mu_\alpha \mathbf{H}_\alpha, & \text{in } \mathbf{R}^3, \\ \nabla \times \mathbf{H}_\alpha = -i\omega\varepsilon_\alpha \mathbf{E}_\alpha + \mathbf{J}_s, & \text{in } \mathbf{R}^3, \\ \nu \times \mathbf{E}_\alpha \quad \text{and } \nu \times \mathbf{H}_\alpha & \text{are continuous across } \partial(z_j + \alpha B_j), \\ \displaystyle\lim_{|x|\to\infty} |x| \left[\nabla \times \mathbf{E}_\alpha - ik\frac{x}{|x|} \times \mathbf{E}_\alpha \right] = 0 & \text{and } \displaystyle\lim_{|x|\to\infty} |x| \left[\nabla \times \mathbf{H}_\alpha - ik\frac{x}{|x|} \times \mathbf{H}_\alpha \right] = 0. \end{cases} \tag{3}$$

Here ν stands for the outgoing unit normal to $\partial(z_j + \alpha B_j)$. We eliminate the magnetic field from the above equations by dividing the first equation in (3) by μ_α and taking the curl. We obtain a problem for $\mathbf{E}_\alpha$:

$$\begin{cases} \nabla \times \mu_\alpha^{-1} \nabla \times \mathbf{E}_\alpha - \omega^2 \varepsilon_\alpha \mathbf{E}_\alpha = i\omega \mathbf{J}_s, & \text{in } \mathbf{R}^3, \\ \nu \times \mathbf{E}_\alpha & \text{is continuous across } \partial(z_j + \alpha B_j), \\ \displaystyle\lim_{|x|\to\infty} |x| \left[\nabla \times \mathbf{E}_\alpha - ik\frac{x}{|x|} \times \mathbf{E}_\alpha \right] = 0. \end{cases} \tag{4}$$

We obtain the magnetic field $\mathbf{H}_\alpha$ from $\mathbf{E}_\alpha$ through the formula, $\mathbf{H}_\alpha = \frac{1}{i\omega\mu_\alpha} \nabla \times \mathbf{E}_\alpha$. The electromagnetic energy is defined by:

$$\mathcal{E}_\alpha := \frac{1}{2} \int_{\mathbf{R}^3} \left(\varepsilon_\alpha \left| \mathbf{E}_\alpha(x) \right|^2 + (\mu_\alpha)^{-1} \left| \mathbf{H}_\alpha(x) \right|^2 \right) \mathrm{d}x. \tag{5}$$

The finiteness of the electromagnetic energy $\mathcal{E}_\alpha$ requires that both electric and magnetic field belong to a space of fields with square integrable **curls**:

$$H(\mathbf{curl}; \mathbf{R}^3) := \left\{ \mathbf{a} \in L^2(\mathbf{R}^3)^3;\ \mathbf{curl}\, \mathbf{a} \in L^2(\mathbf{R}^3)^3 \right\}.$$

It can be shown that there exists a unique solution $\mathbf{E}_\alpha \in H(\mathbf{curl}; \mathbf{R}^3)$ to the problem (4), and this solution satisfies the following Lippman–Schwinger integral representation formula:

Lemma 1.1. *Let* $\mathbf{E}_\alpha$ *be the solution of* (4). *Then, the following integral representation formula holds*

$$\mathbf{E}_\alpha(x) = \mathbf{E}_0(x) + \sum_{j=1}^{m} \int_{z_j+\alpha B_j} \left(-i\omega(\mu_j - \mu_0)\nabla' \times \mathcal{G}(x,x') \cdot \mathbf{H}_\alpha(x') + \omega^2\mu_0(\varepsilon_j - \varepsilon_0)\mathcal{G}(x,x') \cdot \mathbf{E}_\alpha(x')\right) \mathrm{d}x'. \tag{6}$$

The 3×3 matrix valued function $\mathcal{G}$ means the Green's function solution to

$$\begin{cases} \nabla \times \nabla \times \mathcal{G}(x,x') - k^2\mathcal{G}(x,x') = \mathcal{I}_3\delta(x-x'), & \text{in } \mathbf{R}^3, \\ \lim\limits_{|x|\to\infty} |x| \left[\nabla \times \mathcal{G}(x,x') - ik\dfrac{x}{|x|} \times \mathcal{G}(x,x')\right] = 0, \end{cases}$$

where $\mathcal{I}_3$ is the 3×3 identity matrix. In the above notation the curl operator, $\nabla\times$, acts on matrices column by column and $\nabla'\times$ denotes the curl operator with respect to the second variable x'.

The proof of Lemma 1.1 consists of multiplying the first equation in (4) by $\mathcal{G}(x,x') \cdot \mathbf{v}$ ($\mathbf{v} \in \mathbf{R}^3$) and integrating by parts over the domain Ω which contains the set of inclusions $\{z_j + \alpha B_j\}_{j=1}^m$.

2. Asymptotic behavior

As mentioned in the introduction, our analysis will involve the polarization tensors $M \in \mathbf{R}^{3\times3}$. We recall that these tensors are defined by

$$M(q_j/q_0; B_j) := \int_{B_j} \nabla v^q(x)\,\mathrm{d}x, \tag{7}$$

where $\{q_j\}$ (resp. q_0) denote either the set $\{\varepsilon_j\}$ or $\{\mu_j\}$ for $1 \leqslant j \leqslant m$ (resp. denote either ε_0 or μ_0) and where the functions v^q, which depend on the contrast q_j/q_0, are solutions of the following problem,

$$\begin{cases} \nabla \cdot q(x)\nabla v^q(x) = 0, & \text{in } \mathbf{R}^3, \\ v^q(x) - x \to 0 & \text{as } x \to \infty, \end{cases} \tag{8}$$

where

$$q(x) = \begin{cases} q_0, & x \in \mathbf{R}^3 \backslash \overline{B_j}, \\ q_j, & x \in B_j. \end{cases}$$

The following proposition holds:

Proposition 2.1. *Suppose that* (1) *and* (7) *are satisfied and let* $(\mathbf{E}_\alpha, \mathbf{H}_\alpha)$ *be the solution of* (3), *then we have*

(i) *The electric field* $\mathbf{E}_\alpha$ *satisfies the following uniform expansion*

$$\begin{aligned} \mathbf{E}_\alpha(x) = \mathbf{E}_0(x) + \alpha^3 \sum_{j=1}^{m} \big[&-i\omega(\mu_j - \mu_0)\nabla' \times \mathcal{G}(x,z_j) \cdot \mathbf{M}(\mu_j/\mu_0; B_j)\mathbf{H}_0(z_j) \\ &+ \omega^2\mu_0(\varepsilon_j - \varepsilon_0)\mathcal{G}(x,z_j) \cdot \mathbf{M}(\varepsilon_j/\varepsilon_0; B_j)\mathbf{E}_0(z_j)\big] + \mathrm{O}(\alpha^4). \end{aligned} \tag{9}$$

(ii) *The magnetic field* $\mathbf{H}_\alpha$ *satisfies the following uniform expansion*

$$\begin{aligned} \mathbf{H}_\alpha(x) = \mathbf{H}_0(x) + \alpha^3 \sum_{j=1}^{m} \big[&i\omega(\varepsilon_j - \varepsilon_0)\nabla' \times \mathcal{G}(x,z_j) \cdot \mathbf{M}(\varepsilon_j/\varepsilon_0; B_j)\mathbf{E}_0(z_j) \\ &- \omega^2\varepsilon_0(\mu_j - \mu_0)\mathcal{G}(x,z_j) \cdot \mathbf{M}(\mu_j/\mu_0; B_j)\mathbf{H}_0(z_j)\big] + \mathrm{O}(\alpha^4). \end{aligned} \tag{10}$$

Proof. For simplicity, we may focus on the case of a single inhomogeneity, i.e. $m = 1$. The derivation of the asymptotic expansions for any fixed number m of well separated inhomogeneities follows by iteration of the arguments that we will present for the case $m = 1$. In order to simplify the notations we assume that the single inhomogeneity has the form $z + \alpha B$ and denote the electric permittivity and magnetic permeability inside $z + \alpha B$ by ε_* and μ_*, respectively. For the problem (3) stated in section 1 the asymptotic expansion of the solution $(\mathbf{E}_\alpha, \mathbf{H}_\alpha)$ which is uniformly valid in space is constructed by the **method of matched asymptotic expansions** for α small, as done in [3,7,10]. To reveal the nature of the perturbations in the electric and magnetic fields, we use the local variable $\xi = \frac{x-z}{\alpha}$ and we set the field $e_\alpha(\xi) = \mathbf{E}_\alpha(z + \alpha\xi)$ and $h_\alpha(\xi) = \mathbf{H}_\alpha(z + \alpha\xi)$. We expect that $\mathbf{E}_\alpha(x)$ and $\mathbf{H}_\alpha(x)$ will appreciably differ from $\mathbf{E}_0(x)$ and $\mathbf{H}_0(x)$, respectively for x near z, but will be close to $\mathbf{E}_0(x)$ and $\mathbf{H}_0(x)$ for x far from z. Therefore, in the spirit of matched asymptotic expansions, we shall represent each of the fields $\mathbf{E}_\alpha$ and $\mathbf{H}_\alpha$ by two different expansions, an **inner expansion** for x near z, and an **outer expansion** for x far from z. The outer expansion is:

$$\mathbf{E}_\alpha(x) = \mathbf{E}_0(x) + \alpha^{\tau_1} E_1(x) + \alpha^{\tau_2} E_2(x) + \cdots, \qquad \mathbf{H}_\alpha(x) = \mathbf{H}_0(x) + \alpha^{\tau_1} H_1(x) + \alpha^{\tau_2} H_2(x) + \cdots \tag{11}$$

for $|x - z| \gg \alpha$, where $0 < \tau_1 < \tau_2 < \cdots$, and $(E_j, H_j)_{j\geqslant 1}$ are to be found.

We write the inner expansions as

$$\begin{aligned} \mathbf{E}_\alpha(z + \alpha\xi) &= \mathbf{e}_\alpha(\xi) = \mathbf{e}_0(\xi) + \alpha e_1(\xi) + \alpha^2 e_2(\xi) + \cdots, \quad \text{for } |\xi| = \mathrm{O}(1), \\ \mathbf{H}_\alpha(z + \alpha\xi) &= \mathbf{h}_\alpha(\xi) = \mathbf{h}_0(\xi) + \alpha h_1(\xi) + \alpha^2 h_2(\xi) + \cdots, \quad \text{for } |\xi| = \mathrm{O}(1), \end{aligned} \tag{12}$$

where $(e_j, h_j)_{j\geqslant 1}$ are to be found.

We try to equate the inner and the outer expansions in some '**transition region**', as done in [3], within which the stretched variable ξ is large and $x - z$ is small. In this domain the matching conditions are

$$\begin{aligned} &\mathbf{E}_0(x) + \alpha^{\tau_1} E_1(x) + \alpha^{\tau_2} E_2(x) + \cdots \sim \mathbf{e}_0(\xi) + \alpha e_1(\xi) + \alpha^2 e_2(\xi) + \cdots, \\ &\mathbf{H}_0(x) + \alpha^{\tau_1} H_1(x) + \alpha^{\tau_2} H_2(x) + \cdots \sim \mathbf{h}_0(\xi) + \alpha h_1(\xi) + \alpha^2 h_2(\xi) + \cdots. \end{aligned} \tag{13}$$

Observing that

$$\varepsilon_\alpha\left(\frac{x-z}{\alpha}\right) \equiv \varepsilon_0, \quad \text{and} \quad \mu_\alpha\left(\frac{x-z}{\alpha}\right) \equiv \mu_0$$

for $|x - z| \gg \alpha$, we find that the stretched coefficients $\varepsilon(\xi)$ and $\mu(\xi)$ can be defined by

$$\varepsilon(\xi) = \begin{cases} \varepsilon_0, & \xi \in \mathbf{R}^3 \setminus \bar{B}, \\ \varepsilon_*, & \xi \in B, \end{cases} \quad \text{and} \quad \mu(\xi) = \begin{cases} \mu_0, & \xi \in \mathbf{R}^3 \setminus \bar{B}, \\ \mu_*, & \xi \in B. \end{cases}$$

Inserting relation (12) into the Maxwell's equations (3) and formally equating coefficients of α^{-1}, we get from (13) that

$$\begin{cases} \nabla \times \mathbf{e}_0(\xi) = 0, & \text{in } \mathbf{R}^3, \\ \nabla \cdot (\varepsilon(\xi)\mathbf{e}_0(\xi)) = 0, & \text{in } \mathbf{R}^3, \\ \mathbf{e}_0(\xi) \to \mathbf{E}_0(z), & \text{as } |\xi| \to +\infty, \end{cases} \tag{14}$$

and

$$\begin{cases} \nabla \times \mathbf{h}_0(\xi) = 0, & \text{in } \mathbf{R}^3, \\ \nabla \cdot (\mu(\xi)\mathbf{h}_0(\xi)) = 0, & \text{in } \mathbf{R}^3, \\ \mathbf{h}_0(\xi) \to \mathbf{H}_0(z), & \text{as } |\xi| \to +\infty. \end{cases} \tag{15}$$

Therefore,

$$\mathbf{e}_0(\xi) = \nabla v^\varepsilon(\xi)\big(\mathbf{E}_0(z)\big), \qquad \mathbf{h}_0(\xi) = \nabla v^\mu(\xi)\big(\mathbf{H}_0(z)\big),$$

where the function v^q, for $q(\xi) = \varepsilon(\xi)$ or $\mu(\xi)$, is defined in (8).

To finish, we use the definition (7) and we insert the inner expansions into the Lippman–Schwinger integral representation formula (6) for the field $\mathbf{E}_\alpha$ to find (9). The proof of (10) can be deduced similarly. □

To evaluate the influence of the presence of the inhomogeneities on the electromagnetic energy defined by (5), we introduce its density per unit of volume which is denoted by $\aleph_\alpha$. According to Poynting's theorem, the energy density $\aleph_\alpha$ is given by:

$$\aleph_\alpha = \nabla \cdot \mathbf{\Pi}_\alpha + \mathbf{J}_s \cdot \mathbf{E}_\alpha, \tag{16}$$

where $\mathbf{\Pi}_\alpha$ is Poynting's vector,

$$\mathbf{\Pi}_\alpha = \frac{\mathbf{E}_\alpha \times \mathbf{H}_\alpha}{\mu_0}. \tag{17}$$

The following theorem justifies the behavior of the electromagnetic energy in the entire space.

Theorem 2.2. *Suppose that* (1) *and* (16) *are satisfied and suppose that the inclusion B_j is a ball for each $j \in \{1, \dots, m\}$. Then, the following uniform expansion holds*

$$\begin{aligned}\aleph_\alpha(x) - \aleph_0(x) = \alpha^3 \sum_{j=1}^m 3|B_j| \bigg(&\frac{\varepsilon_j - \varepsilon_0}{\varepsilon_j + 2\varepsilon_0} \big[k^2 \mathbf{J}_s \cdot \big(\mathcal{G}(x, z_j) \cdot \mathbf{E}_0(z_j)\big) \\ &+ \frac{1}{\mu_0} \nabla \cdot \big[i\omega\varepsilon_0 \mathbf{E}_0 \times \big(\nabla' \times \mathcal{G}(x, z_j) \cdot \mathbf{E}_0(z_j)\big) + k^2 \big(\mathcal{G}(x, z_j) \cdot \mathbf{E}_0(z_j)\big) \times \mathbf{H}_0(z_j)\big]\big] \\ &- \frac{\mu_j - \mu_0}{\mu_j + 2\mu_0} \big[i\omega\mu_0 \mathbf{J}_s \cdot \big(\nabla' \times \mathcal{G}(x, z_j) \cdot \mathbf{H}_0(z_j)\big) + \frac{1}{\mu_0} \nabla \cdot \big[i\omega\mu_0 \big(\nabla' \times \mathcal{G}(x, z_j) \cdot \mathbf{H}_0(z_j)\big) \\ &+ k^2 \mathbf{E}_0 \times \mathcal{G}(x, z_j) \cdot \mathbf{H}_0(z_j)\big]\big] \bigg) + \mathrm{O}(\alpha^5).\end{aligned}$$

Proof. Recall the following identity: $\nabla \cdot (A \times B) = B \cdot \nabla \times A - A \cdot \nabla \times B$. Relation (17) immediately gives

$$\nabla \cdot (\mathbf{\Pi}_\alpha) = \frac{1}{\mu_0} [\mathbf{H}_\alpha \cdot \nabla \times \mathbf{E}_\alpha - \mathbf{E}_\alpha \cdot \nabla \times \mathbf{H}_\alpha]. \tag{18}$$

Next, under the assumption that inclusion B_j is a ball for all $j \in \{1, \dots, m\}$, it was proved in [6] that polarization tensors (7) are simplified

$$M(\mu_j/\mu_0; B_j) = \frac{3\mu_0}{\mu_j + 2\mu_0} |B_j| \mathcal{I}_3, \quad \text{and} \quad M(\varepsilon_j/\varepsilon_0; B_j) = \frac{3\varepsilon_0}{\varepsilon_j + 2\varepsilon_0} |B_j| \mathcal{I}_3. \tag{19}$$

Then, according to [5] the correction of order four in (9) is zero and the remainder is in fact $\mathrm{O}(\alpha^5)$. Using this result and inserting (19) into (9), one obtain the following expansions:

$$\begin{aligned}\mathbf{E}_\alpha(x) = \mathbf{E}_0(x) + \alpha^3 \sum_{j=1}^m \bigg[&-i\omega\mu_0 \frac{3(\mu_j - \mu_0)}{\mu_j + 2\mu_0} |B_j| \nabla' \times \mathcal{G}(x, z_j) \cdot \mathbf{H}_0(z_j) \\ &+ k^2 \frac{3(\varepsilon_j - \varepsilon_0)}{\varepsilon_j + 2\varepsilon_0} |B_j| \mathcal{G}(x, z_j) \cdot \mathbf{E}_0(z_j) \bigg] + \mathrm{O}(\alpha^5).\end{aligned} \tag{20}$$

In a similar fashion, we obtain the expansion for the magnetic field:

$$\begin{aligned}\mathbf{H}_\alpha(x) = \mathbf{H}_0(x) + \alpha^3 \sum_{j=1}^m \bigg[&i\omega\varepsilon_0 \frac{3(\varepsilon_j - \varepsilon_0)}{\varepsilon_j + 2\varepsilon_0} |B_j| \nabla' \times \mathcal{G}(x, z_j) \cdot \mathbf{E}_0(z_j) \\ &- k^2 \frac{3(\mu_j - \mu_0)}{\mu_j + 2\mu_0} |B_j| \mathcal{G}(x, z_j) \cdot \mathbf{H}_0(z_j) \bigg] + \mathrm{O}(\alpha^5).\end{aligned} \tag{21}$$

Using relations (20) and (21), the following holds:

$$\mathbf{H}_\alpha \cdot \nabla \times \mathbf{E}_\alpha = \mathbf{H}_0 \cdot \nabla \times \mathbf{E}_0 + \alpha^3 \left\{ \sum_{j=1}^m c_1 \mathbf{H}_0 \cdot \nabla \times \big(\nabla' \times \mathcal{G}(x, z_j)\mathbf{H}_0\big) + c_2 \mathbf{H}_0 \cdot \nabla \times \big(\mathcal{G}(x, z_j)\mathbf{E}_0\big) + \sum_{j=1}^m c_1' \big(\nabla' \times \mathcal{G}(x, z_j)\mathbf{E}_0\big) \cdot (\nabla \times \mathbf{E}_0) + c_2' \big(\mathcal{G}(x, z_j)\mathbf{H}_0\big) \cdot (\nabla \times \mathbf{E}_0) \right\} + \mathrm{O}(\alpha^5),$$

where the constants $c_1,\ c_2,\ c_1'$ and c_2' are given by

$$\begin{cases} \dfrac{c_1}{i\omega\mu_0} = \dfrac{c_2'}{k^2} = -3\dfrac{\mu_j - \mu_0}{\mu_j + 2\mu_0}|B_j|, \\ \dfrac{c_2}{k^2} = \dfrac{c_1'}{i\omega\varepsilon_0} = 3\dfrac{\varepsilon_j - \varepsilon_0}{\varepsilon_j + 2\varepsilon_0}|B_j|. \end{cases}$$

In the same manner we find the expansion of the term $\mathbf{E}_\alpha \cdot \nabla \times \mathbf{H}_\alpha$; therefore (18) becomes

$$\nabla \cdot (\mathbf{\Pi}_\alpha) = \nabla \cdot (\mathbf{\Pi}_0) + \alpha^3 \sum_{j=1}^m \frac{1}{\mu_0}\Big[c_1 \nabla \cdot \Big(\big(\nabla' \times \mathcal{G}(x, z_j)\mathbf{H}_0\big) \times \mathbf{H}_0\Big) + c_2 \nabla \cdot \big(\mathcal{G}(x, z_j)\mathbf{E}_0 \times \mathbf{H}_0\big) + c_1' \nabla \cdot \Big(\mathbf{E}_0 \times \big(\nabla' \times \mathcal{G}(x, z_j)\mathbf{E}_0\big)\Big) + c_2' \nabla \cdot \big(\mathbf{E}_0 \times \mathcal{G}(x, z_j)\mathbf{H}_0\big)\Big] + \mathrm{O}(\alpha^5).$$

The proof is achieved by expanding the term $\mathbf{J}_s \cdot \mathbf{E}_\alpha$ in (16) at order 5 with respect to α. □

By using Theorem 2.2, we can prove the main result:

Theorem 2.3. *Suppose that* (1) *is satisfied and suppose that the inclusion B_j is a ball for each $j \in \{1, \ldots, m\}$. Then, there exists a positive constant C such that the following estimate holds*:

$$|\mathcal{E}_\alpha - \mathcal{E}_0| \leqslant C\alpha^3,$$

as $\alpha \to 0$. The constant C is independent of α and the set of points $\{z_j\}_{j=1}^m$, but this constant C depends on $|B_j|$.

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