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An improvement of the Erdős–Turán theorem on the distribution of zeros of polynomials

Tamás Erdélyi

Department of Mathematics, Texas A&M University, College Station, Texas 77843, USA

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Abstract

We prove a subtle 'one-sided' improvement of a classical result of P. Erdős and P. Turán on the distribution of zeros of polynomials. The proof of this improvement is quite short and rather elementary. Nevertheless it allows us to obtain a beautiful recent result of V. Totik and P. Varjú as a simple corollary, and in a somewhat stronger form, without any use of a potential theoretic machinery. Namely, if the modulus of a monic polynomial P of degree n (with complex coefficients) on the unit circle of the complex plane is at most $1 + \mathrm{o}(1)$ uniformly, then the multiplicity of each zero of P on the unit circle is $\mathrm{o}(n^{1/2})$. Our approach is based on the interesting observation that the Erdős–Turán Theorem improves itself. ***To cite this article: T. Erdélyi, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***
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Résumé

Un raffinement du théorème de Erdős–Turán sur la distribution des zéros de polynômes. Nous prouvons un raffinement délicat d'un résultat classique de P. Erdős et P. Turán sur la distribution des zéros de polynômes. Bien que notre preuve soit brève et plutôt élémentaire, elle nous permet d'obtenir comme corollaire et sans recourir à la théorie du potentiel, une amélioration d'un résultat récent et élégant de V. Totik et P. Varjú : si le module sur le cercle unité du plan complexe d'un polynôme P monique, de degré n et à coefficients complexes est uniformément au plus $1 + \mathrm{o}(1)$, alors la multiplicité de chaque zéro de P sur le cercle unité est $\mathrm{o}(n^{1/2})$. Notre approche repose sur l'observation, à notre avis intéressante, que le théorème de Erdős–Turán peut en quelque sorte s'auto-raffiner. ***Pour citer cet article : T. Erdélyi, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***
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0. Introduction

Let ∂D denote the unit circle of the complex plane. Let

$$\|P\| := \max_{z \in \partial D} \big|P(z)\big|.$$

A classical result of Erdős and Turán [3] is the following:

E-mail address: terdelyi@math.tamu.edu.

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Theorem *(Erdős–Turán). If the zeros of*

$$P(z) := \sum_{j=0}^{n} a_j z^j, \quad a_j \in \mathbb{C},\ a_0 a_n \neq 0,$$

are denoted by

$$z_j = r_j \exp(\mathrm{i}\varphi_j), \quad r_j > 0,\ \varphi_j \in [0, 2\pi),\ j = 1, 2, \ldots, n,$$

then for every $0 \leqslant \alpha < \beta \leqslant 2\pi$ *we have*

$$\left| \sum_{j \in I(\alpha,\beta)} 1 - \frac{\beta - \alpha}{2\pi} n \right| < 16\sqrt{n \log R},$$

where

$$R := |a_0 a_n|^{-1/2} \|P\| \quad \textit{and} \quad I(\alpha, \beta) := \{j\colon \alpha \leqslant \varphi_j \leqslant \beta\}.$$

The review [6] written by V. Totik on the recent book [1] of V.V. Andrievskii and H.-P. Blatt shows the central role of the Erdős–Turán Theorem in certain types of investigations.

Note that some books quote this result with

$$R := |a_0 a_n|^{-1/2} \big(|a_0| + |a_1| + \cdots + |a_n|\big)$$

in place of $R := |a_0 a_n|^{-1/2} \|P\|$. In fact, the weaker result is an obvious corollary of the stronger one by observing that $\|P\| \leqslant |a_0| + |a_1| + \cdots + |a_n|$.

1. New result

In this Note we offer a subtle 'one-sided' improvement of the above Erdős–Turán Theorem.

Theorem 1.1. *If the zeros of*

$$P(z) := \sum_{j=0}^{n} a_j z^j, \quad a_j \in \mathbb{C},\ a_0 a_n \neq 0,$$

are denoted by

$$z_j = r_j \exp(\mathrm{i}\varphi_j), \quad r_j > 0,\ \varphi_j \in [0, 2\pi),\ j = 1, 2, \ldots, n,$$

then for every $0 \leqslant \alpha < \beta \leqslant 2\pi$ *we have*

$$\sum_{j \in I_1(\alpha,\beta)} 1 - \frac{\beta - \alpha}{2\pi} n \leqslant 16\sqrt{n \log R_1}, \quad \textit{and} \quad \sum_{j \in I_2(\alpha,\beta)} 1 - \frac{\beta - \alpha}{2\pi} n \leqslant 16\sqrt{n \log R_2},$$

where

$$R_1 := |a_n|^{-1} \|P\|, \qquad R_2 := |a_0|^{-1} \|P\|,$$

and

$$I_1(\alpha, \beta) := \{j\colon \alpha \leqslant \varphi_j \leqslant \beta, r_j \geqslant 1\}, \qquad I_2(\alpha, \beta) := \{j\colon \alpha \leqslant \varphi_j \leqslant \beta, r_j \leqslant 1\}.$$

This result is closely related to a recent paper of V. Totik and P. Varjú [7]. In fact, it may as well be derived from part (ii) of Theorem 1 in [7]. However, here we do not rely on this recent result. Our approach is based on the interesting observation that the Erdős–Turán Theorem above improves itself.

Proof. It is sufficient to prove only the first inequality for

$$P(z) := \sum_{j=0}^{n} a_j z^j, \quad a_j \in \mathbb{C},\ a_0 a_n \neq 0.$$

If we apply it to

$$P^*(z) := z^n \overline{P(1/\bar{z})} = \sum_{j=0}^{n} \bar{a}_{n-j} z^j,$$

we obtain the second inequality of the theorem. Without loss of generality we may also assume that $a_n = 1$. Let

$$k := \sum_{j \in I_1(\alpha,\beta)} 1 - \frac{\beta - \alpha}{2\pi} n.$$

Without loss of generality we may assume that $r_j = 1$ for each $j \in I_1(\alpha, \beta)$. Indeed, if we replace each $z_j = r_j \exp(\mathrm{i}\varphi_j)$ with $r_j > 1$ with $\tilde{z}_j := \exp(\mathrm{i}\varphi_j)$, and introduce

$$\widetilde{P}(z) = \prod_{j \in I_2(\alpha,\beta)} (z - z_j) \prod_{j \in I_1(\alpha,\beta)} (z - \tilde{z}_j) = \sum_{j=0}^{n} \tilde{a}_j z^j,$$

then the polynomial $\widetilde{P}$ is still monic and $\|\widetilde{P}\| \leqslant \|P\|$. Without loss of generality we may also assume that $k \geqslant 0$, otherwise the proof is trivial. We need to prove that $\|P\| \geqslant \exp(\frac{k^2}{256n})$. If $|a_0| \geqslant 1$, then the result follows from the Erdős–Turán Theorem. Suppose $|a_0| < 1$. Let m be a nonnegative integer. Let $S := P^m$. Since each zero of P with multiplicity u on the unit circle is a zero of $S + S^*$ with multiplicity at least mu, the Erdős–Turán Theorem gives

$$\|S + S^*\| > |1 + a_0^m| \exp\left(\frac{(mk)^2}{256mn}\right),$$

hence

$$2\|P\|^m > |1 + a_0^m| \exp\left(\frac{mk^2}{256n}\right).$$

Therefore

$$\|P\| \geqslant 2^{-1/m} \big|1 - |a_0|^m\big|^{1/m} \exp\left(\frac{k^2}{256n}\right) \geqslant 2^{-1/m} \big|1 - |a_0|^m\big| \exp\left(\frac{k^2}{256n}\right)$$

follows. Now let m tend to ∞. We obtain

$$\|P\| \geqslant \exp\left(\frac{k^2}{256n}\right),$$

and the theorem follows. □

As a corollary we obtain a recent result of V. Totik and P. Varjú [7]:

Corollary 1.2. *If the modulus of a monic polynomial P of degree n (with complex coefficients) on the unit circle of the complex plane is at most $1 + \mathrm{o}(1)$ uniformly, then the multiplicity of each zero of P on the unit circle is $\mathrm{o}(n^{1/2})$.*

Actually we get this stronger form of the Totik–Varjú result:

Corollary 1.3.

(i) *If the modulus of a monic polynomial P of degree n (with complex coefficients) on the unit circle of the complex plane is at most $1 + \mathrm{o}(1)$ uniformly, then the multiplicity of each zero of P outside the open unit disk is $\mathrm{o}(n^{1/2})$.*

(ii) *Equivalently, if a complex polynomial P of degree n and constant term 1 has modulus at most $1 + \mathrm{o}(1)$ uniformly on the unit circle, then the multiplicity of each zero of P in the closed unit disk is $\mathrm{o}(n^{1/2})$.*

Proof. Suppose P is a monic polynomial of degree n with k zeros at a point z_0 outside the open unit disk. Then Theorem 1.1 implies $\|P\| \geqslant \exp(\frac{k^2}{256n})$, and the corollary follows. □

We note that Corollary 1.3 extends to generalized polynomials of the form

$$f(z)=\prod_{j=1}^{k}(z-z_j)^{r_j}, \quad z_j\in\mathbb{C},\ r_j>0,$$

where $N:=\sum_{j=1}^{k} r_j$ is the degree of the generalized polynomial.

2. Remarks

A construction of G. Halász [4] shows that for every $k\in\mathbb{N}$, there exists a polynomial h of degree k (with real coefficients) such that $h(0)=1$, $h(1)=0$, and $|h(z)|<\exp(\frac{2}{k})$ for all $z\in\partial D$. (See also [5] concerning the exact constant.) This implies the following observation:

Remark 2.1. $o(n^{1/2})$ in Corollary 1.2 cannot be improved.

Proof. Indeed, let the polynomial h of degree k be picked by the above lemma. Then $H:=h^m$ is a polynomial of degree km, $H(0)=1$, and H has a zero at 1 with multiplicity at least m. Also $|H(z)|<\exp(2m/k)$ for all $z\in\partial D$. Now let Q be defined by $Q(z):=z^{km}H(1/z)$. Then Q is a monic polynomial of degree km and Q has a zero at 1 with multiplicity at least m. Also $|Q(z)|<\exp(2m/k)$ for all $z\in\partial D$. Now choose $m=o(k)$. □

Remark 2.2. A result closely related to Theorem 1.1 is the theorem below. See Theorem 4.1 in [2]. However, Theorem 1.1 does not follow from this.

Theorem *(Borwein–Erdélyi–Kós). There is an absolute constant $c>0$ such that every polynomial p of the form*

$$p(x)=\sum_{j=0}^{n}a_jx^j, \quad |a_j|\leqslant 1,\ |a_0|=1,\ a_j\in\mathbb{C},$$

has at most $c\sqrt{n}$ zeros in $[-1,1]$.

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There is an absolute constant $c>0$ such that every polynomial p of the form

$$p(x)=\sum_{j=0}^{n}a_jx^j, \quad |a_j|\leqslant 1,\ |a_0|=|a_n|=1,\ a_j\in\mathbb{C},$$

has at most $c\sqrt{n}$ real zeros.

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