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Number Theory/Logic

The external fundamental group of an algebraic number field

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Abstract

We associate to every algebraic number field $K/\mathbb{Q}$ a hyperbolic surface lamination and an *external fundamental group* ${}^{\circ}\mathbb{F}_K$: a generalization of the fundamental germ construction of Gendron that necessarily contains external (not first order definable) elements. The external fundamental group ${}^{\circ}\mathbb{F}_{\mathbb{Q}}$ is an extension of the absolute Galois group $\hat{\mathbb{F}}_{\mathbb{Q}}$, that conjecturally contains a subgroup whose abelianization is isomorphic to the idèle class group. ***To cite this article: T.M. Gendron, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Le groupe fondamental externe d'un corps de nombres algébriques. On associe à chaque corps de nombres algébriques $K/\mathbb{Q}$ une lamination en surfaces hyperboliques et un *groupe fondamental externe* ${}^{\circ}\mathbb{F}_K$: une généralisation de la construction du germe fondamental de Gendron, qui contient nécessairement des éléments externes (non definissables au premier ordre). Le groupe fondamental externe ${}^{\circ}\mathbb{F}_{\mathbb{Q}}$ est une extension décomposée du groupe de Galois absolu $\hat{\mathbb{F}}_{\mathbb{Q}}$, qui contient d'après une conjecture un sous groupe avec une « abelianisation » isomorphe au groupe de classes des idèles. ***Pour citer cet article : T.M. Gendron, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

1. Introduction

The search for a geometrization of an algebraic number field $K/\mathbb{Q}$ has been one of the longstanding ambitions of algebraic number theory: indeed, it could be said that the specter of such a geometrization haunts some of its most celebrated enterprises viz. the Riemann hypothesis, non-Abelian class field theory, Grothendieck–Teichmüller theory. One phenomenon which could achieve structural clarity via geometrization is the isomorphism of class field theory $C_{\mathbb{Q}} \cong \mathbb{R}_+^{\times} \times \hat{\mathbb{Z}}^{\times}$, where $C_{\mathbb{Q}}$ is the idèle class group of $\mathbb{Q}$. Since $\hat{\mathbb{Z}}^{\times} \cong \hat{\mathbb{F}}_{\mathbb{Q}}^{\mathrm{ab}}$, where $\hat{\mathbb{F}}_{\mathbb{Q}} = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, it has been suggested by a number of authors [6,1,2] that the factor $\mathbb{R}_+^{\times}$ ought to have also a Galois interpretation. Formally, one seeks an extension $\bar{\mathbb{F}}_{\mathbb{Q}} \to \hat{\mathbb{F}}_{\mathbb{Q}}$ in which $\bar{\mathbb{F}}_{\mathbb{Q}}$ has arithmetic meaning (a "cosmic Galois group" [1]), and for which

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$\bar{\Gamma}_{\mathbb{Q}}^{\mathrm{ab}} \cong C_{\mathbb{Q}}$. In this Note, we shall construct a candidate for $\bar{\Gamma}_{\mathbb{Q}}$, defined as the external fundamental group of a geometrization of $\mathbb{Q}$ by a hyperbolic surface lamination.

2. Internal fundamental group

Let M be a compact n-manifold, $p : \tilde{M} \to M$ a universal cover and write $\pi = \pi_1(M)$. Fix an ultrafilter $\mathfrak{U}$ on $\mathbb{N}$ whose elements are of infinite cardinality. Denote by ${}^*\pi$ the ultraproduct of π with respect to $\mathfrak{U}$. Note that there is a monomorphism $c : \pi \hookrightarrow {}^*\pi$ given by the constant sequences, and we identify π with its image. The ultraproduct ${}^*\pi$ is an example of a non-standard model of π [5].

Suppose that M is Riemannian, and equip $\tilde{M}$ with the pull-back metric so that π acts by isometries on $\tilde{M}$. Let ${}^\bullet\tilde{M}$ be the quotient of ${}^*\tilde{M}$ (= the ultraproduct of $\tilde{M}$) obtained by identifying sequence classes that are asymptotic. There is a canonical surjective map ${}^\bullet\tilde{M} \to M$ which associates to each class ${}^\bullet\tilde{x}$ the limit of $p(\tilde{x}_i)$ where $\{\tilde{x}_i\} \in {}^\bullet\tilde{x}$ is any representative sequence for which $p(\tilde{x}_i)$ converges. Note that ${}^*\pi$ acts on the left on ${}^\bullet\tilde{M}$ and ${}^*\pi\backslash{}^\bullet\tilde{M} \approx M$.

We may view ${}^\bullet\tilde{M}$ as a lamination with discrete transversals: the leaf containing ${}^\bullet\tilde{x} \in {}^\bullet\tilde{M}$ consists of those sequence classes of bounded distance from ${}^\bullet\tilde{x}$, itself a Riemannian manifold. In fact, ${}^\bullet\tilde{M}$ may be identified with the suspension of the inclusion c, i.e. $(\tilde{M} \times {}^*\pi)/\pi$, where $(\tilde{x}, {}^*\alpha) \cdot \gamma = (\gamma \cdot \tilde{x}, ({}^*\alpha)\gamma^{-1})$ for all $\gamma \in \pi$. In the suspension description, the action of ${}^*\pi$ is induced by $(\tilde{x}, {}^*\alpha) \mapsto (\tilde{x}, {}^*\gamma{}^*\alpha)$, ${}^*\gamma \in {}^*\pi$, and so can be seen to be by leaf-wise isometries. This discussion applies to any group extension $\pi \subset G$ (particularly, when G = a non-standard model of π), the appropriate universal covering space being the suspension of the inclusion $\pi \hookrightarrow G$.

We now indicate how ${}^*\pi$ codifies laminated coverings of M. For simplicity, we shall restrict ourselves to suspensions over M. Let G be a compact topological group and let $\rho : \pi \to G$ be a representation. The suspension of ρ, denoted $M(\rho)$, is a principal G-bundle as well as a lamination over M, minimal if and only if ρ has dense image, with simply connected leaves if and only if $\mathrm{Ker}(\rho) = 1$. Three examples:

a. If $G = 1$ then $M(\rho) \approx M$.
b. Let $G = \hat{\pi}$ = the profinite completion of π, ρ the canonical map. Then $M(\rho) \approx \hat{M}$ = the algebraic universal cover of M, a $\hat{\pi}$-principal bundle over M e.g. $\hat{\pi}\backslash\hat{M} \approx M$. It is classical that $\hat{M}$ and $\hat{\pi}$ are the appropriate notions of universal cover and fundamental group for M within the étale category.
c. Let $M = G = \mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$, and for $r \in \mathbb{R} - \mathbb{Q}$, define ρ by $\rho(n) = \overline{nr}$ = the image of nr in $\mathbb{S}^1$. Then $M(\rho) = \mathcal{F}_r$ = the irrational foliation of the 2-torus by lines of slope r.

An analogue of the fundamental group for $M(\rho)$ is given by the *fundamental germ* $[\![\pi]\!] = [\![\pi]\!]_1 M(\rho)$, [3,4]. In the case when the suspension $M(\rho)$ is minimal, it has the following description. Since ρ has dense image, the 'standard part' map $\mathrm{std}(\rho) : {}^*\pi \to G$, defined by taking a sequence class to the unique limit in G of its image by ρ, is onto. We define $[\![\pi]\!] := \mathrm{Ker}(\mathrm{std}(\rho))$ and refer to $1 \to [\![\pi]\!] \to {}^*\pi \to G \to 1$ as the *standardization exact sequence*. For the three examples above we have:

a. $[\![\pi]\!]_1 M = {}^*\pi$.
b. $[\![\pi]\!]_1 \hat{M} = \bigcap {}^*H$ where $H < \pi$ runs through the subgroups of finite index. This is a non-trivial subgroup of ${}^*\pi$ even when π is residually finite i.e when $\bigcap H$ is trivial (for example, when M is a compact surface).
c. We say that a sequence class ${}^*\epsilon \in {}^*\mathbb{R}$ is an infinitesimal if it contains a sequence converging to 0. Then we may identify $[\![\pi]\!]_1 \mathcal{F}_r$ with the subgroup of ${}^*n \in {}^*\mathbb{Z}$ for which $r{}^*n + {}^*m$ is an infinitesimal for some ${}^*m \in {}^*\mathbb{Z}$: in other words, $[\![\pi]\!]_1 \mathcal{F}_r$ is the group of Diophantine approximations of r.

We now discuss covering space theory. Let $M(\rho)$ be as above, assumed for simplicity to be minimal with simply connected leaves. Assume also that M has been equipped with a Riemannian metric, so that $M(\rho)$ has a leaf-wise Riemannian metric. There is a canonical map $\tilde{M} \to M(\rho)$, induced by $\tilde{M} \times \{1\} \hookrightarrow \tilde{M} \times G$. The image of this map is a leaf L_0 called the canonical leaf. There is a surjective map ${}^\bullet\tilde{M} \to M(\rho)$ – assigning to a sequence class ${}^\bullet\tilde{x}$ the limit of its image via $\tilde{M} \to M(\rho)$ – which is a local isometry along the leaves. Any continuous self-map of $M(\rho)$ preserving L_0 lifts uniquely to a self-map of ${}^\bullet\tilde{M}$. The natural action of $[\![\pi]\!]$ on ${}^\bullet\tilde{M}$ has quotient $[\![\pi]\!]\backslash{}^\bullet\tilde{M}$ which is in canonical bijection with $M(\rho)$. For example, when $M = \Gamma\backslash\mathbb{H}^2$ is a closed hyperbolic surface, we may identify $[\![\pi]\!]_1 M(\rho)$ with a 'Fuchsian germ' $[\![\Gamma]\!] < \mathrm{PSL}(2, {}^*\mathbb{R})$ and $[\![\Gamma]\!]\backslash{}^\bullet\mathbb{H}^2$ is in bijection with $M(\rho)$.

It is possible to endow ${}^{\bullet}\tilde{M}$ with a non-trivial transverse topology in such a way that $[\![\pi]\!]$ acts by homeomorphisms and so that the quotient $[\![\pi]\!]\backslash{}^{\bullet}\tilde{M}$ is homeomorphic to $M(\rho)$. To do this, we choose a set-theoretic section $s : G \to {}^{*}\pi$ of std(ρ), so that $s(\rho(\gamma)) = \gamma$ for all $\gamma \in \pi$, and for which $s(G)$ is a right π-set. Then if we give ${}^{*}\pi$ the topology: (topology of G) $\times$ (discrete), this gives a topology on $\tilde{M} \times {}^{*}\pi$ invariant by the action of π, hence inducing a topology on ${}^{\bullet}\tilde{M}$. The left multiplication action by elements of $[\![\pi]\!]$ permutes the "cosets" $[\![x]\!]s(G)$, $[\![x]\!] \in [\![\pi]\!]$, hence $[\![\pi]\!]$ acts by homeomorphisms, and with the quotient topology, the bijection between $[\![\pi]\!]\backslash{}^{\bullet}\tilde{M}$ and $M(\rho)$ is a homeomorphism. (N.B. We may even choose the section s in order that any leaf of ${}^{\bullet}\tilde{M}$ intersects a given $s(G)$-transversal no more than once: so that ${}^{\bullet}\tilde{M}$ is a lamination with no non-trivial holonomy.)

3. External fundamental group

Let F be the free group on two generators, $\hat{F}$ its profinite completion and consider the standardization sequence $1 \to [\![F]\!] \to {}^{*}F \to \hat{F} \to 1$. Neither ${}^{*}F$ nor $\hat{F}$ are free groups in the discrete (combinatorial) sense. Let $\hat{\mathbf{F}}$ be the free group generated by $\hat{F}$ (viewed as a set), which has cardinality of the continuum. By universality, there is a canonical epimorphism $\hat{p} : \hat{\mathbf{F}} \to \hat{F}$. If $\sigma : \hat{F} \hookrightarrow {}^{*}F$ is a set-theoretic section of the standardization sequence whose image contains a generating set of ${}^{*}F$, then the induced map ${}^{*}p : \hat{\mathbf{F}} \to {}^{*}F$ is an epimorphism, and $\hat{p} = \mathrm{std} \circ {}^{*}p$ (by the uniqueness part of universality). If $\hat{K}, {}^{*}K$ are the kernels of $\hat{p}, {}^{*}p$ then ${}^{*}K < \hat{K}$.

Denote by $\mathrm{Aut}(\hat{F})$ the group of bicontinuous automorphisms of $\hat{F}$, and by ${}^{\circ}\mathrm{Aut}(F)$ the subgroup of $\mathrm{Aut}({}^{*}F)$ of automorphisms which induce elements of $\mathrm{Aut}(\hat{F})$ i.e. automorphisms which stabilize $[\![F]\!]$ and induce bicontinuous automorphisms of $\hat{F}$. Note that $\mathrm{Aut}(F)$ as well as ${}^{*}\mathrm{Aut}(F)$ include canonically in ${}^{\circ}\mathrm{Aut}(F)$. Indeed, if ${}^{*}A \in {}^{*}\mathrm{Aut}(F)$ and ${}^{*}x \in [\![F]\!]$, then ${}^{*}A({}^{*}x)$ is represented by a sequence $\{A_i(x_i)\}$, and $A_i(x_i)$ is in a subgroup of index $N_i \to \infty$ if and only if x_i is.

Theorem 3.1. *The canonical homomorphism* ${}^{\circ}\mathrm{Aut}(F) \to \mathrm{Aut}(\hat{F})$ *is surjective.*

The theorem is proved as follows: note first that any element $\alpha \in \mathrm{Aut}(\hat{F})$ defines a bijection of the generating set of $\hat{\mathbf{F}}$, hence an automorphism $\boldsymbol{\alpha}$ of the latter. As such, $\boldsymbol{\alpha}$ necessarily stabilizes $\hat{K}$: we may arrange that it also stabilizes ${}^{*}K$ by composing, if necessary, with a suitable automorphism covering the identity of $\hat{F}$. The result descends to an automorphism ${}^{\circ}\alpha$ of ${}^{*}F$. The association $\alpha \mapsto {}^{\circ}\alpha$ evidently defines a (set-theoretic) section.

Denote by ${}^{\circ}\mathrm{Inn}(F)$ those elements of ${}^{\circ}\mathrm{Aut}(F)$ which map to inner automorphisms of $\hat{F}$. (N.B. ${}^{*}F$, acting innerly, is a subgroup of ${}^{\circ}\mathrm{Inn}(F)$.) If we denote by ${}^{\circ}\mathrm{Out}(F)$ the quotient of ${}^{\circ}\mathrm{Aut}(F)$ by ${}^{\circ}\mathrm{Inn}(F)$, we obtain an exact sequence $1 \to [\![\Gamma]\!] \to {}^{\circ}\mathrm{Out}(F) \to \mathrm{Out}(\hat{F}) \to 1$.

It is important to note that ${}^{\circ}\mathrm{Out}(F)$ contains as a *proper* subgroup the ultraproduct ${}^{*}\mathrm{Out}(F) \cong {}^{*}\mathrm{GL}(2,\mathbb{Z}) \cong \mathrm{GL}(2,{}^{*}\mathbb{Z})$. The latter is called the group of *internal* outer automorphisms of ${}^{*}F$, and elements of ${}^{\circ}\mathrm{Out}(F)$ which are not internal are called *external*. That we cannot replace ${}^{\circ}\mathrm{Out}(F)$ by ${}^{*}\mathrm{Out}(F)$ is borne out by the following:

Fact 1. Although F is dense in $\hat{F}$, $\mathrm{Out}(F)$ is not dense in $\mathrm{Out}(\hat{F})$, hence $\mathrm{Out}(\hat{F})$ is not the profinite completion of $\mathrm{Out}(F) \cong \mathrm{GL}(2,\mathbb{Z})$. Thus, ${}^{*}\mathrm{Out}(F)$ does not map epimorphically onto $\mathrm{Out}(\hat{F})$.

Recall that the theory of a group G is the collection $\mathrm{Th}(G)$ of all first order sentences which are true in G. We say G' is a non-standard model of G if $\mathrm{Th}(G') = \mathrm{Th}(G)$ but $G' \not\cong G$. For example, the ultrapower ${}^{*}G$ is a non-standard model of G.

Question 1. Is ${}^{\circ}\mathrm{Out}(F)$ a non-standard model of $\mathrm{Out}(F)$?

In what follows $K/\mathbb{Q}$ is an arbitrary algebraic number field and $\hat{\Gamma}_K$ is its absolute Galois group. Recall the Belyi monomorphism $\beta : \hat{\Gamma}_K \subset \hat{\Gamma}_{\mathbb{Q}} \hookrightarrow \mathrm{Out}(\hat{F})$. We will not distinguish between $\hat{\Gamma}_K$ and its image in $\mathrm{Out}(\hat{F})$. Let $\mathrm{SL}(2,\mathbb{Z}) \cong \mathrm{Out}_{+}(F) \hookrightarrow \mathrm{Out}(\hat{F})$ be the canonical inclusion. Define $\hat{\Sigma} = \hat{\Sigma}_{\bar{\mathbb{Q}}}$ as the suspension $(\mathbb{H}^2 \times \mathrm{Out}(\hat{F}))/\mathrm{SL}(2,\mathbb{Z})$, where the action of $A \in \mathrm{SL}(2,\mathbb{Z})$ is defined $A(z,f) = (Az, fA^{-1})$. Then $\hat{\Sigma}$ is a non-minimal solenoid by hyperbolic surface orbifolds that covers the modular orbifold $\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}^2$. The action of $\hat{\Gamma}_K$ on the product $\mathbb{H}^2 \times \mathrm{Out}(\hat{F})$, $\hat{\sigma}(z,f) = (z, \hat{\sigma}f)$, descends to an action on $\hat{\Sigma}$ by leaf-wise isometries. Since $\hat{\Gamma}_K$ is a closed subgroup of $\mathrm{Out}(\hat{F})$, the quotient $\hat{\Sigma}_K = \hat{\mathbb{\Gamma}}_K \backslash \hat{\Sigma}$ is also a lamination by hyperbolic surface orbifolds. By construction, the association $K \mapsto \hat{\Sigma}_K$ is Galois natural.

Denote by ${}^{\circ}\mathbb{\Gamma}_K$ the pre-image of $\hat{\mathbb{\Gamma}}_K$ in ${}^{\circ}\mathrm{Out}(F)$ so that $[\![\mathbb{\Gamma}]\!]$ is the kernel of the standardization epimorphism ${}^{\circ}\mathbb{\Gamma}_K \to \hat{\mathbb{\Gamma}}_K$. We have $[\![\mathbb{\Gamma}]\!] = \bigcap {}^{\circ}\mathbb{\Gamma}_K$. Recall that there is a canonical inclusion $\mathrm{SL}(2,\mathbb{Z}) \cong \mathrm{Out}_+(F) \hookrightarrow {}^{\circ}\mathrm{Out}(F)$. By suspending this inclusion with respect to the action of $\mathrm{SL}(2,\mathbb{Z})$ on $\mathbb{H}^2$, we obtain a trivial lamination which we denote ${}^{\circ}\mathbb{H}^2$. We note that the quotient of ${}^{\circ}\mathbb{H}^2$ by the left action of ${}^{\circ}\mathrm{Out}(F)$ is isometric to the modular orbifold.

We topologize ${}^{\circ}\mathrm{Out}(F)$ by choosing a set-theoretic section of ${}^{\circ}\mathrm{Out}(F) \to \mathrm{Out}(\hat{F})$ whose image is a right $\mathrm{SL}(2,\mathbb{Z})$-set and which maps $\mathrm{SL}(2,\mathbb{Z})$ to itself (as we did at the end of the last section). This induces a topology on ${}^{\circ}\mathbb{H}^2$ making it a solenoid by hyperbolic surface orbifolds, with respect to which the action by ${}^{\circ}\mathrm{Out}(F)$ is by homeomorphisms which are isometries along the leaves. The quotient by $[\![\mathbb{\Gamma}]\!]$ can be identified with $\hat{\Sigma} = \hat{\Sigma}_{\bar{\mathbb{Q}}}$ and in addition $\hat{\Sigma}_K \cong {}^{\circ}\mathbb{\Gamma}_K \backslash {}^{\circ}\mathbb{H}^2 \cong \hat{\mathbb{\Gamma}}_K \backslash \hat{\Sigma}$. This justifies viewing ${}^{\circ}\mathbb{\Gamma}_K$ as a fundamental group, in a way which generalizes the internal fundamental group defined in §2.

Conjecture 3.2. *There is a subgroup $\bar{\mathbb{\Gamma}}_{\mathbb{Q}} < {}^{\circ}\mathbb{\Gamma}_{\mathbb{Q}}$ which is an extension of $\hat{\mathbb{\Gamma}}_{\mathbb{Q}}$ and for which $\bar{\mathbb{\Gamma}}_{\mathbb{Q}}^{\mathrm{ab}} \cong C_{\mathbb{Q}}$.*

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References

[1] P. Cartier, La folle journée, de Grothendieck à Connes et Kontsevich : évolution des notions d'espace et de symétrie, in : Les relations entre les mathématiques et la physique théorique, Inst. Hautes Études Sci., Bures-sur-Yvette, 1998, pp. 23–42.

[2] A. Connes, Trace formula in noncommutative geometry and the zeros of the Riemann zeta function, Selecta Math. (N.S.) 5 (1) (1999) 29–106.

[3] T.M. Gendron, The algebraic theory of the fundamental germ, Bull. Braz. Math. Soc. (N.S.) 37 (1) (2006) 49–87.

[4] T.M. Gendron, The geometric theory of the fundamental germ, Nagoya Math. J., in press.

[5] A. Robinson, Non-Standard Analysis, Princeton Landmarks in Mathematics, Princeton University Press, Princeton, NJ, 1996.

[6] A. Weil, Sur la théorie du corps de classes, J. Math. Soc. Japan 3 (1951) 1–35.