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Mathematical Analysis

On some new applications of power increasing sequences

Hüseyin Bor

Department of Mathematics, Erciyes University, 38039 Kayseri, Turkey Received 17 August 2007; accepted after revision 29 January 2008 Available online 10 March 2008 Presented by Jean-Pierre Kahane

Abstract

In the present Note, a result dealing with $|\bar{N}, p_n|_k$ summability has been generalized for $|\bar{N}, p_n, \theta_n|_k$ summability factors under more weaker conditions. Also some new results have obtained. To cite this article: H. Bor, C. R. Acad. Sci. Paris, Ser. I 346 (2008).

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Résumé

Sur quelques nouvelles applications des puissances des suites croissantes. Dans cette Note, nous donnons une généralisation d'un résultat connu de $|\bar{N}, p_n|_k$ sommabilité portant sur les facteurs de summabilité $|\bar{N}, p_n, \theta_n|_k$ sous des hypothèses plus faibles. Nous obtenons des résultats nouveaux. *Pour citer cet article : H. Bor, C. R. Acad. Sci. Paris, Ser. I 346 (2008).* © 2008 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

1. Introduction

We denote by $\mathcal{BV}_{\mathcal{O}}$ the expression $\mathcal{BV} \cap \mathcal{C}_{\mathcal{O}}$, where $\mathcal{C}_{\mathcal{O}}$ and \mathcal{BV} are the set of all null sequences and the set of all sequences with bounded variation, respectively. Let $\sum a_n$ be a given infinite series with partial sums (s_n) . Let (p_n) be a sequence of positive numbers such that $P_n = p_0 + p_1 + \cdots + p_n \to \infty$ as $n \to \infty$. The series $\sum a_n$ is said to be summable $|\bar{N}, p_n|_k, k \ge 1$, if (see [1])

$$\sum_{n=1}^{\infty} (P_n/p_n)^{k-1} |\sigma_n - \sigma_{n-1}|^k < \infty,$$
(1)

where

$$\sigma_n = \frac{1}{P_n} \sum_{\nu=0}^n p_\nu s_\nu.$$
⁽²⁾

Let (θ_n) be any sequence of positive real constants. The series $\sum a_n$ is said to be summable $|\bar{N}, p_n, \theta_n|_k, k \ge 1$, if (see [6])

E-mail address: bor@erciyes.edu.tr.

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$$\sum_{n=1}^{\infty} \theta_n^{k-1} |\sigma_n - \sigma_{n-1}|^k < \infty.$$
(3)

If we take $\theta_n = \frac{p_n}{p_n}$, then $|\bar{N}, p_n, \theta_n|_k$ summability reduces to $|\bar{N}, p_n|_k$ summability. Also if we take $\theta_n = n$ and $p_n = 1$ for all values of n, then we get $|C, 1|_k$ (see [4]) summability.

Furthermore if we take $\theta_n = n$, then $|\bar{N}, p_n, \theta_n|_k$ summability reduces to $|R, p_n|_k$ (see [2]) summability.

2. Known result

Mazhar [5] has proved the following theorem dealing with $|\bar{N}, p_n|_k$ summability factors by using an almost increasing sequence:

Theorem A. Let (X_n) be an almost increasing sequence and let there be sequences (λ_n) and (p_n) such that

$$|\lambda_m|X_m = \mathcal{O}(1) \quad as \ m \to \infty, \tag{4}$$

$$\sum_{n=1}^{m} nX_n |\Delta^2 \lambda_n| = \mathcal{O}(1), \tag{5}$$

$$\sum_{n=1}^{m} \frac{P_n}{n} = \mathcal{O}(P_m) \quad as \ m \to \infty.$$
(6)

If

$$\sum_{n=1}^{m} \frac{|t_n|^k}{n} = \mathcal{O}(X_m) \quad as \ m \to \infty,$$

$$\sum_{m=1}^{m} \frac{p_n}{n} = \mathcal{O}(X_m) \quad as \ m \to \infty,$$
(7)

$$\sum_{n=1}^{\infty} \frac{P_n}{P_n} |t_n|^k = \mathcal{O}(X_m) \quad as \ m \to \infty,$$
(8)

then the series $\sum a_n \lambda_n$ is summable $|\bar{N}, p_n|_k, k \ge 1$.

3. The main result

In the present Note, we make use of the concept of a quasi β -power increasing sequence in order to generalize Theorem A for the $|\bar{N}, p_n, \theta_n|_k$ summability under considerably weaker conditions. Now we shall prove the following theorem:

Theorem. Let $(\lambda_n) \in \mathcal{BV}_{\mathcal{O}}$ and (X_n) be a quasi β -power increasing sequence for some $0 < \beta < 1$. If $(\frac{\theta_n p_n}{P_n})$ is a non-increasing sequence and the conditions (4)–(7),

$$\sum_{n=1}^{m} \theta_n^{k-1} \left(\frac{p_n}{P_n}\right)^k |t_n|^k = \mathcal{O}(X_m) \quad as \ m \to \infty,$$
(9)

are satisfied, then the series $\sum a_n \lambda_n$ is summable $|\bar{N}, p_n, \theta_n|_k, k \ge 1$.

It should be noted that if we take (X_n) as an almost increasing sequence and $\theta_n = \frac{P_n}{p_n}$, then we obtain Theorem A. In this case condition (9) reduces to condition (8), the condition $(\frac{\theta_n p_n}{P_n})$ which is a non-increasing sequence automatically satisfied and the condition $(\lambda_n) \in \mathcal{BV}_{\mathcal{O}}$ is not needed.

We need following lemma for the proof of our theorem:

Lemma. (See [3].) Under the conditions of the theorem, we have that

392

$$nX_n |\Delta\lambda_n| = O(1) \quad as \ n \to \infty, \tag{10}$$

$$\sum_{n=1}^{\infty} X_n |\Delta\lambda_n| < \infty. \tag{11}$$

4. Proof of the theorem

Let (T_n) denotes the (\overline{N}, p_n) mean of the series $\sum a_n \lambda_n$. Then, for $n \ge 1$, we have

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^n P_{v-1} a_v \lambda_v = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^n \frac{P_{v-1} \lambda_v}{v} v a_v.$$

By Abel's transformation, we have

$$T_n - T_{n-1} = \frac{n+1}{nP_n} p_n t_n \lambda_n - \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} p_v t_v \lambda_v \frac{v+1}{v} + \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v \Delta \lambda_v t_v \frac{v+1}{v} + \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v t_v \lambda_{v+1} \frac{1}{v} = T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}, \quad \text{say.}$$

To complete the proof of the theorem, by Minkowski's inequality, it is enough to show that

$$\sum_{n=1}^{\infty} \theta_n^{k-1} |T_{n,r}|^k < \infty, \quad \text{for } r = 1, 2, 3, 4.$$

Firstly, applying Abel's transformation we have that

$$\sum_{n=1}^{m} \theta_n^{k-1} |T_{n,1}|^k = \mathcal{O}(1) \sum_{n=1}^{m} |\lambda_n| \theta_n^{k-1} \left(\frac{p_n}{P_n}\right)^k |t_n|^k = \mathcal{O}(1) \sum_{n=1}^{m-1} |\Delta \lambda_n| X_n + \mathcal{O}(1) |\lambda_m| X_m = \mathcal{O}(1) \quad \text{as } m \to \infty,$$

by virtue of the hypotheses of the theorem and lemma. Now, when k > 1 applying Hölder's inequality, as in $T_{n,1}$, we have that

$$\sum_{n=2}^{m+1} \theta_n^{k-1} |T_{n,2}|^k = O(1) \sum_{\nu=1}^m p_\nu |\lambda_\nu|^{k-1} |\lambda_\nu| |t_\nu|^k \sum_{n=\nu+1}^{m+1} \left(\frac{\theta_n p_n}{P_n}\right)^{k-1} \frac{p_n}{P_n P_{n-1}}$$
$$= O(1) \sum_{\nu=1}^m \left(\frac{\theta_\nu p_\nu}{P_\nu}\right)^{k-1} p_\nu |t_\nu|^k |\lambda_\nu| \sum_{n=\nu+1}^{m+1} \frac{p_n}{P_n P_{n-1}}$$
$$= O(1) \sum_{\nu=1}^m |\lambda_\nu| \theta_\nu^{k-1} \left(\frac{p_\nu}{P_\nu}\right)^k |t_\nu|^k = O(1) \quad \text{as } m \to \infty.$$

Again in the similar way, we have that

$$\begin{split} \sum_{n=2}^{m+1} \theta_n^{k-1} |T_{n,3}|^k &= O(1) \sum_{v=1}^m P_v |t_v|^k |\Delta\lambda_v| \sum_{n=v+1}^{m+1} \left(\frac{\theta_n p_n}{p_n}\right)^{k-1} \frac{p_n}{P_n P_{n-1}} \\ &= O(1) \sum_{v=1}^m \left(\frac{\theta_v p_v}{P_v}\right)^{k-1} |\Delta\lambda_v| |t_v|^k = O(1) \left(\frac{\theta_1 p_1}{P_1}\right)^{k-1} \sum_{v=1}^m v |\Delta\lambda_v| \frac{1}{v} |t_v|^k \\ &= O(1) \sum_{v=1}^{m-1} v |\Delta^2\lambda_v| X_v + O(1) \sum_{v=1}^{m-1} |\Delta\lambda_v| X_v + O(1)m |\Delta\lambda_m| X_m = O(1) \quad \text{as } m \to \infty, \end{split}$$

by virtue of the hypotheses of the theorem and lemma. Finally, as in $T_{n,1}$ we have that

$$\begin{split} \sum_{n=2}^{m+1} \theta_n^{k-1} |T_{n,4}|^k &= \mathcal{O}(1) \sum_{v=1}^m P_v |\lambda_{v+1}| |t_v|^k \frac{1}{v} \sum_{n=v+1}^{m+1} \left(\frac{\theta_n p_n}{P_n}\right)^{k-1} \frac{p_n}{P_n P_{n-1}} \\ &= \mathcal{O}(1) \sum_{v=1}^m |\lambda_{v+1}| \left(\frac{\theta_v p_v}{P_v}\right)^{k-1} \frac{|t_v|^k}{v} = \mathcal{O}(1) \left(\frac{\theta_1 p_1}{P_1}\right)^{k-1} \sum_{v=1}^m |\lambda_{v+1}| \frac{|t_v|^k}{v} \\ &= \mathcal{O}(1) \sum_{v=1}^{m-1} |\Delta \lambda_{v+1}| X_{v+1} + \mathcal{O}(1) |\lambda_{m+1}| X_{m+1} = \mathcal{O}(1) \quad \text{as } m \to \infty, \end{split}$$

in view of the hypotheses of the theorem and lemma. This completes the proof of the theorem. If we take $p_n = 1$ for all values of n and $\theta_n = n$, then we get a result dealing with $|C, 1|_k$ summability factors. Also if we take $p_n = 1$ for all values of n, then we have a new result for $|C, 1, \theta_n|_k$ summability. Finally if we take $\theta_n = n$, then we have an another new result for $|R, p_n|_k$ summability.

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