## Mathematical Analysis

# On some new applications of power increasing sequences 

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#### Abstract

In the present Note, a result dealing with $\left|\bar{N}, p_{n}\right|_{k}$ summability has been generalized for $\left|\bar{N}, p_{n}, \theta_{n}\right|_{k}$ summability factors under more weaker conditions. Also some new results have obtained. To cite this article: H. Bor, C. R. Acad. Sci. Paris, Ser. I 346 (2008).


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## Résumé

Sur quelques nouvelles applications des puissances des suites croissantes. Dans cette Note, nous donnons une généralisation d'un résultat connu de $\left|\bar{N}, p_{n}\right|_{k}$ sommabilité portant sur les facteurs de summabilité $\left|\bar{N}, p_{n}, \theta_{n}\right|_{k}$ sous des hypothèses plus faibles. Nous obtenons des résultats nouveaux. Pour citer cet article : H. Bor, C. R. Acad. Sci. Paris, Ser. I 346 (2008). © 2008 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

## 1. Introduction

We denote by $\mathcal{B} \mathcal{V}_{\mathcal{O}}$ the expression $\mathcal{B} \mathcal{V} \cap \mathcal{C}_{\mathcal{O}}$, where $\mathcal{C}_{\mathcal{O}}$ and $\mathcal{B} \mathcal{V}$ are the set of all null sequences and the set of all sequences with bounded variation, respectively. Let $\sum a_{n}$ be a given infinite series with partial sums $\left(s_{n}\right)$. Let ( $p_{n}$ ) be a sequence of positive numbers such that $P_{n}=p_{0}+p_{1}+\cdots+p_{n} \rightarrow \infty$ as $n \rightarrow \infty$. The series $\sum a_{n}$ is said to be summable $\left|\bar{N}, p_{n}\right|_{k}, k \geqslant 1$, if (see [1])

$$
\begin{equation*}
\sum_{n=1}^{\infty}\left(P_{n} / p_{n}\right)^{k-1}\left|\sigma_{n}-\sigma_{n-1}\right|^{k}<\infty \tag{1}
\end{equation*}
$$

where

$$
\begin{equation*}
\sigma_{n}=\frac{1}{P_{n}} \sum_{v=0}^{n} p_{v} s_{v} \tag{2}
\end{equation*}
$$

Let $\left(\theta_{n}\right)$ be any sequence of positive real constants. The series $\sum a_{n}$ is said to be summable $\left|\bar{N}, p_{n}, \theta_{n}\right|_{k}, k \geqslant 1$, if (see [6])

[^0]\[

$$
\begin{equation*}
\sum_{n=1}^{\infty} \theta_{n}^{k-1}\left|\sigma_{n}-\sigma_{n-1}\right|^{k}<\infty \tag{3}
\end{equation*}
$$

\]

If we take $\theta_{n}=\frac{P_{n}}{p_{n}}$, then $\left|\bar{N}, p_{n}, \theta_{n}\right|_{k}$ summability reduces to $\left|\bar{N}, p_{n}\right|_{k}$ summability. Also if we take $\theta_{n}=n$ and $p_{n}=1$ for all values of $n$, then we get $|C, 1|_{k}$ (see [4]) summability.

Furthermore if we take $\theta_{n}=n$, then $\left|\bar{N}, p_{n}, \theta_{n}\right|_{k}$ summability reduces to $\left|R, p_{n}\right|_{k}$ (see [2]) summability.

## 2. Known result

Mazhar [5] has proved the following theorem dealing with $\left|\bar{N}, p_{n}\right|_{k}$ summability factors by using an almost increasing sequence:

Theorem A. Let $\left(X_{n}\right)$ be an almost increasing sequence and let there be sequences $\left(\lambda_{n}\right)$ and $\left(p_{n}\right)$ such that

$$
\begin{align*}
& \left|\lambda_{m}\right| X_{m}=\mathrm{O}(1) \quad \text { as } m \rightarrow \infty,  \tag{4}\\
& \sum_{n=1}^{m} n X_{n}\left|\Delta^{2} \lambda_{n}\right|=\mathrm{O}(1),  \tag{5}\\
& \sum_{n=1}^{m} \frac{P_{n}}{n}=\mathrm{O}\left(P_{m}\right) \quad \text { as } m \rightarrow \infty . \tag{6}
\end{align*}
$$

If

$$
\begin{align*}
& \sum_{n=1}^{m} \frac{\left|t_{n}\right|^{k}}{n}=\mathrm{O}\left(X_{m}\right) \quad \text { as } m \rightarrow \infty  \tag{7}\\
& \sum_{n=1}^{m} \frac{p_{n}}{P_{n}}\left|t_{n}\right|^{k}=\mathrm{O}\left(X_{m}\right) \quad \text { as } m \rightarrow \infty \tag{8}
\end{align*}
$$

then the series $\sum a_{n} \lambda_{n}$ is summable $\left|\bar{N}, p_{n}\right|_{k}, k \geqslant 1$.

## 3. The main result

In the present Note, we make use of the concept of a quasi $\beta$-power increasing sequence in order to generalize Theorem A for the $\left|\bar{N}, p_{n}, \theta_{n}\right|_{k}$ summability under considerably weaker conditions. Now we shall prove the following theorem:

Theorem. Let $\left(\lambda_{n}\right) \in \mathcal{B} \mathcal{V}_{\mathcal{O}}$ and $\left(X_{n}\right)$ be a quasi $\beta$-power increasing sequence for some $0<\beta<1$. If $\left(\frac{\theta_{n} p_{n}}{P_{n}}\right)$ is a non-increasing sequence and the conditions (4)-(7),

$$
\begin{equation*}
\sum_{n=1}^{m} \theta_{n}^{k-1}\left(\frac{p_{n}}{P_{n}}\right)^{k}\left|t_{n}\right|^{k}=\mathrm{O}\left(X_{m}\right) \quad \text { as } m \rightarrow \infty \tag{9}
\end{equation*}
$$

are satisfied, then the series $\sum a_{n} \lambda_{n}$ is summable $\left|\bar{N}, p_{n}, \theta_{n}\right|_{k}, k \geqslant 1$.
It should be noted that if we take $\left(X_{n}\right)$ as an almost increasing sequence and $\theta_{n}=\frac{P_{n}}{p_{n}}$, then we obtain Theorem A. In this case condition (9) reduces to condition (8), the condition ( $\frac{\theta_{n} p_{n}}{P_{n}}$ ) which is a non-increasing sequence automatically satisfied and the condition $\left(\lambda_{n}\right) \in \mathcal{B} \mathcal{V}_{\mathcal{O}}$ is not needed.

We need following lemma for the proof of our theorem:
Lemma. (See [3].) Under the conditions of the theorem, we have that

$$
\begin{align*}
& n X_{n}\left|\Delta \lambda_{n}\right|=\mathrm{O}(1) \quad \text { as } n \rightarrow \infty  \tag{10}\\
& \sum_{n=1}^{\infty} X_{n}\left|\Delta \lambda_{n}\right|<\infty \tag{11}
\end{align*}
$$

## 4. Proof of the theorem

Let $\left(T_{n}\right)$ denotes the $\left(\bar{N}, p_{n}\right)$ mean of the series $\sum a_{n} \lambda_{n}$. Then, for $n \geqslant 1$, we have

$$
T_{n}-T_{n-1}=\frac{p_{n}}{P_{n} P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_{v} \lambda_{v}=\frac{p_{n}}{P_{n} P_{n-1}} \sum_{v=1}^{n} \frac{P_{v-1} \lambda_{v}}{v} v a_{v}
$$

By Abel's transformation, we have

$$
\begin{aligned}
T_{n}-T_{n-1}= & \frac{n+1}{n P_{n}} p_{n} t_{n} \lambda_{n}-\frac{p_{n}}{P_{n} P_{n-1}} \sum_{v=1}^{n-1} p_{v} t_{v} \lambda_{v} \frac{v+1}{v}+\frac{p_{n}}{P_{n} P_{n-1}} \sum_{v=1}^{n-1} P_{v} \Delta \lambda_{v} t_{v} \frac{v+1}{v} \\
& +\frac{p_{n}}{P_{n} P_{n-1}} \sum_{v=1}^{n-1} P_{v} t_{v} \lambda_{v+1} \frac{1}{v}=T_{n, 1}+T_{n, 2}+T_{n, 3}+T_{n, 4}, \quad \text { say. }
\end{aligned}
$$

To complete the proof of the theorem, by Minkowski's inequality, it is enough to show that

$$
\sum_{n=1}^{\infty} \theta_{n}^{k-1}\left|T_{n, r}\right|^{k}<\infty, \quad \text { for } r=1,2,3,4
$$

Firstly, applying Abel's transformation we have that

$$
\sum_{n=1}^{m} \theta_{n}^{k-1}\left|T_{n, 1}\right|^{k}=\mathrm{O}(1) \sum_{n=1}^{m}\left|\lambda_{n}\right| \theta_{n}^{k-1}\left(\frac{p_{n}}{P_{n}}\right)^{k}\left|t_{n}\right|^{k}=\mathrm{O}(1) \sum_{n=1}^{m-1}\left|\Delta \lambda_{n}\right| X_{n}+\mathrm{O}(1)\left|\lambda_{m}\right| X_{m}=\mathrm{O}(1) \quad \text { as } m \rightarrow \infty
$$

by virtue of the hypotheses of the theorem and lemma. Now, when $k>1$ applying Hölder's inequality, as in $T_{n, 1}$, we have that

$$
\begin{aligned}
\sum_{n=2}^{m+1} \theta_{n}^{k-1}\left|T_{n, 2}\right|^{k} & =\mathrm{O}(1) \sum_{v=1}^{m} p_{v}\left|\lambda_{v}\right|^{k-1}\left|\lambda_{v}\right|\left|t_{v}\right|^{k} \sum_{n=v+1}^{m+1}\left(\frac{\theta_{n} p_{n}}{P_{n}}\right)^{k-1} \frac{p_{n}}{P_{n} P_{n-1}} \\
& =\mathrm{O}(1) \sum_{v=1}^{m}\left(\frac{\theta_{v} p_{v}}{P_{v}}\right)^{k-1} p_{v}\left|t_{v}\right|^{k}\left|\lambda_{v}\right| \sum_{n=v+1}^{m+1} \frac{p_{n}}{P_{n} P_{n-1}} \\
& =\mathrm{O}(1) \sum_{v=1}^{m}\left|\lambda_{v}\right| \theta_{v}^{k-1}\left(\frac{p_{v}}{P_{v}}\right)^{k}\left|t_{v}\right|^{k}=\mathrm{O}(1) \quad \text { as } m \rightarrow \infty .
\end{aligned}
$$

Again in the similar way, we have that

$$
\begin{aligned}
\sum_{n=2}^{m+1} \theta_{n}^{k-1}\left|T_{n, 3}\right|^{k} & =\mathrm{O}(1) \sum_{v=1}^{m} P_{v}\left|t_{v}\right|^{k}\left|\Delta \lambda_{v}\right| \sum_{n=v+1}^{m+1}\left(\frac{\theta_{n} p_{n}}{p_{n}}\right)^{k-1} \frac{p_{n}}{P_{n} P_{n-1}} \\
& =\mathrm{O}(1) \sum_{v=1}^{m}\left(\frac{\theta_{v} p_{v}}{P_{v}}\right)^{k-1}\left|\Delta \lambda_{v}\right|\left|t_{v}\right|^{k}=\mathrm{O}(1)\left(\frac{\theta_{1} p_{1}}{P_{1}}\right)^{k-1} \sum_{v=1}^{m} v\left|\Delta \lambda_{v}\right| \frac{1}{v}\left|t_{v}\right|^{k} \\
& =\mathrm{O}(1) \sum_{v=1}^{m-1} v\left|\Delta^{2} \lambda_{v}\right| X_{v}+\mathrm{O}(1) \sum_{v=1}^{m-1}\left|\Delta \lambda_{v}\right| X_{v}+\mathrm{O}(1) m\left|\Delta \lambda_{m}\right| X_{m}=\mathrm{O}(1) \quad \text { as } m \rightarrow \infty
\end{aligned}
$$

by virtue of the hypotheses of the theorem and lemma. Finally, as in $T_{n, 1}$ we have that

$$
\begin{aligned}
\sum_{n=2}^{m+1} \theta_{n}^{k-1}\left|T_{n, 4}\right|^{k} & =\mathrm{O}(1) \sum_{v=1}^{m} P_{v}\left|\lambda_{v+1}\right|\left|t_{v}\right|^{k} \frac{1}{v} \sum_{n=v+1}^{m+1}\left(\frac{\theta_{n} p_{n}}{P_{n}}\right)^{k-1} \frac{p_{n}}{P_{n} P_{n-1}} \\
& =\mathrm{O}(1) \sum_{v=1}^{m}\left|\lambda_{v+1}\right|\left(\frac{\theta_{v} p_{v}}{P_{v}}\right)^{k-1} \frac{\left|t_{v}\right|^{k}}{v}=\mathrm{O}(1)\left(\frac{\theta_{1} p_{1}}{P_{1}}\right)^{k-1} \sum_{v=1}^{m}\left|\lambda_{v+1}\right| \frac{\left|t_{v}\right|^{k}}{v} \\
& =\mathrm{O}(1) \sum_{v=1}^{m-1}\left|\Delta \lambda_{v+1}\right| X_{v+1}+\mathrm{O}(1)\left|\lambda_{m+1}\right| X_{m+1}=\mathrm{O}(1) \quad \text { as } m \rightarrow \infty,
\end{aligned}
$$

in view of the hypotheses of the theorem and lemma. This completes the proof of the theorem. If we take $p_{n}=1$ for all values of $n$ and $\theta_{n}=n$, then we get a result dealing with $|C, 1|_{k}$ summability factors. Also if we take $p_{n}=1$ for all values of $n$, then we have a new result for $\left|C, 1, \theta_{n}\right|_{k}$ summability. Finally if we take $\theta_{n}=n$, then we have an another new result for $\left|R, p_{n}\right|_{k}$ summability.

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