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Algebraic Geometry

Atiyah–Drinfeld–Hitchin–Manin construction of framed instanton sheaves

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Abstract

We introduce a generalization of Atiyah–Drinfeld–Hitchin–Manin equation, which is subsequently used to construct a class of sheaves on projective spaces that arise in connection with instanton theory. ***To cite this article: M. Jardim, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Construction de Atiyah–Drinfeld–Hitchin–Manin de faisceaux trivialisés d'instantons. Nous introduisons une généralisation de l'equation de Atiyah–Drinfeld–Hitchin–Manin que nous utilisons ensuite pour construire une classe de faisceaux sur des espaces projectifs que l'on rencontre dans le contexte de la théorie des instantons. ***Pour citer cet article : M. Jardim, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

1. Introduction

A surprising connection between theoretical physics and algebraic geometry appeared in the late 1970s, when Atiyah, Drinfeld, Hitchin and Manin provided a complete classification of instantons on the 4-dimensional sphere S^4 using the Penrose–Ward correspondence between instantons on S^4 and certain holomorphic vector bundles on $\mathbb{P}^3$ together with a characterization of vector bundles on $\mathbb{P}^3$ due to Horrocks [2]. Later, Donaldson noticed in [4] that instantons on S^4 were also in correspondence with framed holomorphic bundles on $\mathbb{P}^2$, while Mamone Capria and Salamon [8] generalized the Penrose–Ward correspondence to a correspondence between quaternionic instantons on $\mathbb{HP}^k$ and certain holomorphic vector bundles on $\mathbb{P}^{2k+1}$. Motivated by these works, Okonek and Spindler introduced the notion of mathematical instanton bundle on $\mathbb{P}^{2k+1}$ [10]; this is a simple, locally-free sheaf E of rank $2k$ on $\mathbb{P}^{2k+1}$ with total Chern class given by $c_t(E) = (1-t^2)^{-c}$ for some $c \in \mathbb{Z}_+$, and natural cohomology in the range $-2k-1 \leqslant p \leqslant 0$. Since then, such objects have been studied by many authors, see for instance [1,3] and the references therein; the main

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questions are whether such bundles are always stable, and the study of its moduli space. More recently, the author has proposed in [7] the following definition:

Definition 1.1. An instanton sheaf on $\mathbb{P}^n$ ($n \geqslant 2$) is a torsion-free coherent sheaf E on $\mathbb{P}^n$ with $c_1(E) = 0$ satisfying the following cohomological conditions:

(i) for $n \geqslant 2$, $H^0(E(-1)) = H^n(E(-n)) = 0$;
(ii) for $n \geqslant 3$, $H^1(E(-2)) = H^{n-1}(E(1-n)) = 0$;
(iii) for $n \geqslant 4$, $H^p(E(k)) = 0$, $2 \leqslant p \leqslant n-2$ and $\forall k$.

The integer $c = h^1(E(-1)) = -\chi(E(-1))$ is called the charge of E.

Recall that a torsion-free sheaf E on $\mathbb{P}^n$ is said to be of trivial splitting type if there is a line $\ell \subset \mathbb{P}^n$ such that the restriction $E|_\ell$ is the free sheaf, i.e. $E|_\ell \simeq \mathcal{O}_\ell^{\oplus \mathrm{rk} E}$. A *framing* on E is the choice of an isomorphism $\phi : E|_\ell \to \mathcal{O}_\ell^{\oplus \mathrm{rk} E}$. A *framed sheaf* is pair (E, ϕ) consisting of a torsion-free sheaf E of trivial splitting type and a framing ϕ.

With these definitions in mind, a mathematical instanton bundle on $\mathbb{P}^{2k+1}$ in the sense of [10] is a rank $2k$ locally-free instanton sheaf on $\mathbb{P}^{2k+1}$ of trivial splitting type (they do not consider framings). Therefore, our definition generalizes [10] by considering more general sheaves of arbitrary rank on projective spaces of arbitrary dimension. Many properties and some explicit examples of instanton sheaves were considered in [7]. The goal of this Note is to present a construction of all framed instanton sheaves on $\mathbb{P}^n$ which generalizes the ADHM construction of framed torsion-free sheaves on $\mathbb{P}^2$ in [4,9] and the construction of framed instanton sheaves on $\mathbb{P}^3$ recently provided in [6]. In this way, we also create a tool to study their moduli spaces, and to address the question of stability.

2. d-dimensional ADHM data

Let V and W be complex vector spaces, with dimensions c and r, respectively, and consider maps $B_1, B_2 \in \mathrm{End}(V)$, $i \in \mathrm{Hom}(W, V)$ and $j \in \mathrm{Hom}(V, W)$. Recall that this so-called *ADHM datum* (B_1, B_2, i, j) is said to be *stable* if there is no proper subspace $S \subset V$ such that $B_k(S) \subset S$ ($k = 1, 2$) and $i(W) \subset S$, and that it is said to be *costable* if there is no proper subspace $S \subset V$ such that $B_k(S) \subset S$ ($k = 1, 2$) and $S \subset \ker j$. Finally, (B_1, B_2, i, j) is *regular* if it is both stable and costable.

Now take $d \geqslant 0$, and consider the following data ($k = 0, \dots, d$ and $l = 1, 2$):

$$B_{kl} \in \mathrm{Hom}(V, V),$$
$$i_k \in \mathrm{Hom}(W, V), \qquad j_k \in \mathrm{Hom}(V, W),$$

and define:

$$\tilde{B}_1 = B_{10}z_0 + \dots + B_{1d}z_d \quad \text{and} \quad \tilde{B}_2 = B_{20}z_0 + \dots + B_{2d}z_d. \tag{1}$$

Thinking of $[z_0 : \dots : z_d]$ as homogeneous coordinates of a projective space $\mathbb{P}^d$, $\tilde{B}_1$ and $\tilde{B}_2$ can be considered as sections of $\mathrm{Hom}(V, V) \otimes \mathcal{O}_{\mathbb{P}^d}(1)$. Define also,

$$\tilde{\imath} = i_0z_0 + \dots + i_dz_d \quad \text{and} \quad \tilde{\jmath} = j_0z_0 + \dots + j_dz_d. \tag{2}$$

Similarly, $\tilde{\imath}$ and $\tilde{\jmath}$ can be regarded as sections of $\mathrm{Hom}(W, V) \otimes \mathcal{O}_{\mathbb{P}^d}(1)$ and $\mathrm{Hom}(V, W) \otimes \mathcal{O}_{\mathbb{P}^d}(1)$, respectively. By the notation $\tilde{B}_1(p)$, $\tilde{B}_2(p)$, $\tilde{\imath}(p)$ and $\tilde{\jmath}(p)$ we mean the evaluation of the sections $\tilde{B}_1$, $\tilde{B}_2$, $\tilde{\imath}$ and $\tilde{\jmath}$ at a point $p \in \mathbb{P}^d$.

Definition 2.1. A d-dimensional ADHM datum (B_{kl}, i_k, j_k) is said to be:

(i) *semistable* if there is $p \in \mathbb{P}^d$ such that $(\tilde{B}_1(p), \tilde{B}_2(p), \tilde{\imath}(p), \tilde{\jmath}(p))$ is stable;
(ii) *stable* if $(\tilde{B}_1(p), \tilde{B}_2(p), \tilde{\imath}(p), \tilde{\jmath}(p))$ is stable for all $p \in \mathbb{P}^d$;
(iii) *semiregular* if it is stable and there is $p \in \mathbb{P}^d$ such that $(\tilde{B}_1(p), \tilde{B}_2(p), \tilde{\imath}(p), \tilde{\jmath}(p))$ is regular;
(iv) *regular* if $(\tilde{B}_1(p), \tilde{B}_2(p), \tilde{\imath}(p), \tilde{\jmath}(p))$ is regular for all $p \in \mathbb{P}^d$.

In this Note, we consider the following generalization of the ADHM equation:

$$[\tilde{B}_1, \tilde{B}_2] + \tilde{\imath}\,\tilde{\jmath} = 0. \tag{3}$$

For $d = 0$, (1) and (2) reduce to the usual ADHM data and (3) reduces to the usual ADHM equation. In general, (3) can be broken down to $\binom{d+2}{2}$ equations involving the homomorphisms B_{kl}, i_k and j_k:

$$[B_{k1}, B_{k2}] + i_k j_k = 0, \quad k = 0, \ldots, d,$$

$$[B_{k1}, B_{m2}] + [B_{k2}, B_{m1}] + i_k j_m + i_m j_k = 0, \quad k < m = 0, \ldots, d.$$

The case $d = 1$ was considered in [4,6] in the context of Yang–Mills theory and the Penrose correspondence; its regular solutions are in 1–1 correspondence with complex instantons on the compactified, complexified Minkowski space-time.

The following existence result can be obtained from our main result (Theorem 3.1 below) and the Main Theorem of [5]:

Proposition 2.2. *Semistable solutions of* (3) *exist for all* $d \geqslant 0$, $r \geqslant 1$ *and* $c \geqslant 1$. *Stable and semiregular solutions of* (3) *exist provided* $r \geqslant d + 1$. *Regular solutions exist provided* $r \geqslant d + 1$ *for d odd and* $r \geqslant d + 2$ *for d even.*

3. Construction of framed instantons sheaves

Let $[z_0 : \cdots : z_d : x : y]$ denote homogeneous coordinates on $\mathbb{P}^{d+2}$, and set $\ell_\infty = \{z_0 = \cdots = z_d = 0\}$, called the *line at infinity*. Our main result is the following:

Theorem 3.1. *There is a* 1–1 *correspondence between stable solutions of the d-dimensional ADHM equation* (3), *and rank r torsion-free instanton sheaves of charge c on* $\mathbb{P}^{d+2}$ *which are framed at* ℓ_∞, *where* $r = \dim W$ *and* $c = \dim V$. *Furthermore, E is reflexive iff the corresponding ADHM datum is semiregular, while E is locally-free iff the corresponding ADHM datum is regular.*

In this section, we describe one way of this correspondence, from stable d-dimensional ADHM data to framed instanton sheaves. The reverse direction follows from the Beilinson spectral sequence [7, Theorem 3] together with a generalization of the argument in [9, Section 2.1]. Let (B_{kl}, i_k, j_k) be a solution of (3); set $n = d + 2$ and $\tilde{W} = V \oplus V \oplus W$. Consider the complex of sheaves:

$$V \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \xrightarrow{\alpha} \tilde{W} \otimes \mathcal{O}_{\mathbb{P}^n} \xrightarrow{\beta} V \otimes \mathcal{O}_{\mathbb{P}^n}(1), \tag{4}$$

where the maps α and β are given by ($\mathbf{1}$ denotes the identity endomorphism of V):

$$\alpha = \begin{pmatrix} \tilde{B}_1 + x\mathbf{1} \\ \tilde{B}_2 + y\mathbf{1} \\ \tilde{\jmath} \end{pmatrix}, \tag{5}$$

$$\beta = (-\tilde{B}_2 - y\mathbf{1} \quad \tilde{B}_1 + x\mathbf{1} \quad \tilde{\imath}). \tag{6}$$

A straightforward calculation shows that $\beta\alpha = 0$ if and only if (B_{kl}, i_k, j_k) satisfies (3).

Proposition 3.2. *Given any d-dimensional ADHM datum* (B_{kl}, i_k, j_k), *the sheaf map* α *is injective, and the variety* $\Sigma = \{X \in \mathbb{P}^n \mid \alpha_X$ *is not injective*$\}$ *has codimension at least* 2. *The sheaf map* β *is surjective if and only if* (B_{kl}, i_k, j_k) *is stable.*

Proof. It easy to see that α_X is injective for all $X \in \ell_\infty$. This means that the localized map α_X may fail to be injective only at a subvariety $\Sigma \subset \mathbb{P}^n$ that does not intersect ℓ_∞, therefore Σ must be of codimension at least 2 and sheaf map α is injective.

Similarly, it is easy to see that β_X is surjective for all $X \in \ell_\infty$. We argue that (B_{kl}, i_k, j_k) is stable if and only if the dual map β_X^* is injective for all $X = [z_0 : \cdots : z_d : x : y] \in \mathbb{P}^n \setminus \ell_\infty$; notice that $[z_0 : \cdots : z_d]$ defines a point $p \in \mathbb{P}^d$. Indeed, if β_X^* is not injective for some $X = [z_0 : \cdots : z_d : x : y]$, then there is $v \in V$ such that

$$\tilde{B}_1(p)^* v = \bar{x} v, \quad \tilde{B}_2(p)^* v = -\bar{y} v, \quad \text{and} \quad \tilde{\imath}(p)^* v = 0, \tag{7}$$

where $p = [z_0 : \cdots : z_d] \in \mathbb{P}^d$. By duality, this implies that $(\tilde{B}_1(p), \tilde{B}_2(p), \tilde{\imath}(p), \tilde{\jmath}(p))$ is not stable, thus (B_{kl}, i_k, j_k) is not stable.

The converse statement is now clear: if (B_{kl}, i_k, j_k) is not stable, then by duality β_X^* is not injective for some $X = [z_0 : \cdots : z_d : x : y]$, hence β cannot be surjective as a sheaf map. □

It follows from Proposition 3.2 that if (B_{kl}, i_k, j_k) is stable, then the complex (4) is actually a linear monad. As it is well known, (see for instance [7, Section 1]), the cohomology $E = \ker\beta/\mathrm{im}\,\alpha$ of the linear monad (4) is a rank r torsion-free instanton sheaf on $\mathbb{P}^n$, of charge c; it is easy to see that $E|_{\ell_\infty} \xrightarrow{\sim} W \otimes \mathcal{O}_{\ell_\infty}$, so that E is framed at ℓ_∞.

If (B_{kl}, i_k, j_k) is not stable, the complex (4) becomes a *framed perverse sheaf*, i.e. an object $E^\bullet$ of the derived category $D^{\mathrm{b}}(\mathbb{P}^n)$ satisfying:

(i) $H^p(E^\bullet) = 0$ for $p \neq 0, 1$;
(ii) $H^0(E^\bullet)$ is a torsion-free sheaf framed at ℓ_∞;
(iii) $H^1(E^\bullet)$ is a torsion sheaf whose support does not intersect ℓ_∞.

In the case at hand, $H^0(E^\bullet) = \ker\beta/\mathrm{im}\,\alpha$ and $H^1(E^\bullet) = \mathrm{coker}\,\beta$.

Finally, it follows from our description that the moduli spaces $\mathcal{F}_{\mathbb{P}^n}(r, c)$ of framed torsion-free instanton sheaves on $\mathbb{P}^n$ of rank r and charge c are given by the set of stable solutions of (3) modulo the following action of GL(V):

$$g \cdot (B_{kl}, i_k, j_k) = (g B_{kl} g^{-1}, g i_k, j_k g^{-1}).$$

Using techniques of Geometric Invariant Theory, one can show that $\mathcal{F}_{\mathbb{P}^n}(r, c)$ is non-empty quasi-projective variety for any values of $n \geqslant 2$, $r \geqslant n - 1$ and $c \geqslant 1$; moreover, $\mathcal{F}_{\mathbb{P}^2}(r, c)$ and $\mathcal{F}_{\mathbb{P}^3}(r, 1)$ are known to be non-singular and irreducible, and to have dimension $2rc$ and $4r$, respectively (see [9] and [6]). In general, it is not known whether $\mathcal{F}_{\mathbb{P}^n}(r, c)$ is either non-singular or irreducible, and how to compute its dimension (compare with [7, Section 5]).

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