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Probability Theory

Rough path integral of local time

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Abstract

In this Note, for a continuous semimartingale local time L_t^x, we establish the integral $\int_{-\infty}^{\infty} g(x)\,\mathrm{d}L_t^x$ as a rough path integral for any finite q-variation function g ($2 \leqslant q < 3$) by using Lyons' rough path integration. We therefore obtain the Tanaka–Meyer formula for a continuous function f if $\nabla^- f$ exists and is of finite q-variation, $2 \leqslant q < 3$. The case when $1 \leqslant q < 2$ was established by Feng and Zhao [C.R. Feng, H.Z. Zhao, Two-parameter p, q-variation path and integration of local times, Potential Analysis 25 (2006) 165–204] using the Young integral. ***To cite this article: C. Feng, H. Zhao, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Intégrale curviligne non régulière du temps local. Dans cette Note, pour un temps local d'une semi-martingale continue, nous définissons l'intégrale $\int_{-\infty}^{\infty} g(x)\,\mathrm{d}L_t^x$ pour toute fonction g de q-variation finie ($2 \leqslant q < 3$) en utilisant l'intégrale de Lyons pour des chemins non-réguliers. Nous obtenons alors la formule de Tanaka–Meyer pour une fonction continue f lorsque $\nabla^- f$ existe et est de q-variation finie avec $2 \leqslant q < 3$. Le cas correspondant à $1 \leqslant q < 2$ utilise l'intégrale de Young (voir Feng et Zhao [C.R. Feng, H.Z. Zhao, Two-parameter p, q-variation path and integration of local times, Potential Analysis 25 (2006) 165–204.]). ***Pour citer cet article : C. Feng, H. Zhao, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Consider a continuous semimartingale $X_t = M_t + V_t$, where M_t is a continuous local martingale and V_t is a continuous process with finite variation. In [1], Lemma 2.1 says that the local time L_t^x of X_t is of bounded p-variation in x for any $t \geqslant 0$ a.s. for any $p > 2$. So in Theorem 2.2, we gave a new condition for the Tanaka–Meyer formula and the integral $\int_{-\infty}^{\infty} \nabla^- f(x)\,\mathrm{d}_x L_t^x$ as a Young integral, when $\nabla^- f(x)$ is of bounded q-variation ($1 \leqslant q < 2$). But what about the case $q \geqslant 2$? Let us first try to define the integral $\int_{-\infty}^{\infty} g(x)\,\mathrm{d}_x L_t^x$ pathwise for a continuous function $g(x)$ with bounded q-variation ($2 \leqslant q < 3$); we also take $2 < p < 3$. We still decompose the local time $L_t^x = \tilde{L}_t^x + \sum_k \hat{L}_t^{x_k^*} 1_{\{x_k^* \leqslant x\}}$, where $\hat{L}_t^x := L_t^x - L_t^{x-}$. Here $\tilde{L}_t^x$ is continuous in x, and $x_k^*, k = 1, 2, \ldots,$ are the countable discontinuous points of L_t^x. From Lemma 2.2 in [1], we know that $h(t,x) := \sum_k \hat{L}_t^{x_k^*} 1_{\{x_k^* \leqslant x\}}$ is of bounded variation in x for each t. So the key

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point is to define $\int_{-\infty}^{\infty} g(x)\,\mathrm{d}_x \tilde{L}_t^x$ pathwise for continuous $g(x)$ with bounded q-variation ($2 \leqslant q < 3$). For this, we will use Lyons' rough path theory, see [3] and also [2].

Let $[x', x'']$ be any interval in R. From the proof of Lemma 2.1 in [1], for any $p \geqslant 2$, we know that there exists a constant $c > 0$ such that $E|\tilde{L}_t^b - \tilde{L}_t^a|^p \leqslant c|b-a|^{p/2}$, i.e. $\tilde{L}_t^x$ satisfies the Hölder condition in [3] with exponent $\frac{1}{2}$. Denote by w a control of $g(x)$. Then $|g(b)-g(a)|^q \leqslant w(a,b)$, for any $(a,b) \in \Delta := \{(a,b)\colon\ x' \leqslant a < b \leqslant x''\}$. It is obvious that $w_1(a,b) := w(a,b) + (b-a)$ is also a control of g. Set $h = \frac{1}{q}$. It is trivial to see that, for any $\theta > q$ (so $h\theta > 1$), $|g(b)-g(a)|^\theta \leqslant w_1(a,b)^{h\theta}$, for any $(a,b) \in \Delta$. Denote $Z_x = (\tilde{L}_t^x, g(x))$. Then we can see that Z_x satisfies, for such $h = \frac{1}{q}$, and any $\theta > q$,

$$E|Z_b - Z_a|^\theta \leqslant c w_1(a,b)^{h\theta} \quad \text{for any } (a,b) \in \Delta, \tag{1}$$

for some constant $c > 0$. Set $w_1(x) := w_1(x', x)$, and for any $m \in N$, $D_m := \{x' = x_0^m < x_1^m < \cdots < x_{2^m}^m = x''\}$ a partition of $[x', x'']$ such that $w_1(x_l^m) - w_1(x_{l-1}^m) = \frac{1}{2^m} w_1(x', x'')$. It is obvious that $x_l^m - x_{l-1}^m \leqslant \frac{1}{2^m} w_1(x', x'')$. Now define a continuous and bounded variation path $Z(m)$ by:

$$Z(m)_x := Z_{x_{l-1}^m} + \frac{w_1(x) - w_1(x_{l-1}^m)}{w_1(x_l^m) - w_1(x_{l-1}^m)} \Delta_l^m Z \quad \text{for } x_{l-1}^m \leqslant x < x_l^m, \tag{2}$$

where $l = 1, \ldots, 2^m$, and $\Delta_l^m Z = Z_{x_l^m} - Z_{x_{l-1}^m}$. The corresponding smooth rough path $\mathbf{Z}(m)$ is built by taking its iterated path integrals, i.e. for any $(a,b) \in \Delta$,

$$\mathbf{Z}(m)_{a,b}^j = \int\limits_{a < x_1 < \cdots < x_j < b} \mathrm{d}Z(m)_{x_1} \otimes \cdots \otimes \mathrm{d}Z(m)_{x_j}. \tag{3}$$

Recall that the θ-variation metric d_θ on $C_{0,\theta}(\Delta, T^{([\theta])}(R^2))$ is defined by [3]:

$$d_\theta(\mathbf{Z}, \mathbf{Y}) = \max_{1 \leqslant i \leqslant [\theta]} d_{i,\theta}\big(\mathbf{Z}^i, \mathbf{Y}^i\big) = \max_{1 \leqslant i \leqslant [\theta]} \sup_{D_{[x',x'']}} \left(\sum_l |\mathbf{Z}_{x_{l-1},x_l}^i - \mathbf{Y}_{x_{l-1},x_l}^i|^{\theta/i} \right)^{i/\theta}.$$

In the following, we will show that $\{\mathbf{Z}(m)\}_{m\in N}$ converges to a geometric rough path $\mathbf{Z}$ in the θ-variation topology when $2 \leqslant q < 3$, in which $\mathbf{Z}_{a,b}^1 = Z_b - Z_a$. We call $\mathbf{Z}$ the canonical geometric rough path associated with Z.

About the first level path $\mathbf{Z}(m)_{a,b}^1$, the method and results are similar to those in Chapter 4 in [3]. We can prove $\sup_m \sup_D \sum_l |\mathbf{Z}(m)_{x_{l-1},x_l}^1|^\theta < \infty$ a.s. and,

Theorem 1. *Assume $q \geqslant 1$. Let $\theta > q$. For the continuous process $Z_x = (\tilde{L}_t^x, g(x))$ satisfying* (1)*, we have*

$$\sum_{m=1}^{\infty} \sup_D \left(\sum_l \big|\mathbf{Z}(m)_{x_{l-1},x_l}^1 - \mathbf{Z}_{x_{l-1},x_l}^1\big|^\theta \right)^{1/\theta} < \infty \quad a.s. \tag{4}$$

In particular, $(\mathbf{Z}(m)_{a,b}^1)$ converges to $(\mathbf{Z}_{a,b}^1)$ in the θ-variation distance a.s. for any $(a,b) \in \Delta$.

We next consider the second level path $\mathbf{Z}(m)_{a,b}^2$. As in [3], we can also see that, if $m \leqslant n$, $\mathbf{Z}(m)_{x_{k-1}^n, x_k^n}^2 = 2^{2(m-n)-1}(\Delta_l^m Z)^{\otimes 2}$, where l is chosen such that $x_{l-1}^m \leqslant x_{k-1}^n < x_k^n \leqslant x_l^m$, and if $m > n$,

$$\mathbf{Z}(m)_{x_{k-1}^n, x_k^n}^2 = \frac{1}{2} \Delta_k^n Z \otimes \Delta_k^n Z + \frac{1}{2} \sum_{\substack{r,l = 2^{m-n}(k-1)+1 \\ r<l}}^{2^{m-n}k} \big(\Delta_r^m Z \otimes \Delta_l^m Z - \Delta_l^m Z \otimes \Delta_r^m Z\big). \tag{5}$$

Similar to the proof of Proposition 4.3.3 in [3], we have:

Proposition 2. *Assume $q \geqslant 2$. Let $\theta > q$. Then for $m \leqslant n$,*

$$\sum_{k=1}^{2^n} E\big|\mathbf{Z}(m+1)_{x_{k-1}^n, x_k^n}^2 - \mathbf{Z}(m)_{x_{k-1}^n, x_k^n}^2\big|^{\theta/2} \leqslant C \left(\frac{1}{2^{n+m}} \right)^{(\theta h - 1)/2}, \tag{6}$$

where C depends on θ, $h(:=\frac{1}{q})$, $w_1(x',x'')$, *and c in* (1).

For the case when $m>n$, we have the following key estimate:

Proposition 3. *Assume* $2\leqslant q<4$. *Let* $q<\theta<4$. *Then for* $m>n$, *we have that*

$$E\big|\mathbf{Z}(m+1)^2_{x^n_{k-1},x^n_k}-\mathbf{Z}(m)^2_{x^n_{k-1},x^n_k}\big|^{\theta/2}\leqslant C\bigg[\bigg(\frac{1}{2^n}\bigg)^{\theta/4}\bigg(\frac{1}{2^m}\bigg)^{\frac{1}{2}h\theta}+\bigg(\frac{1}{2^n}\bigg)^{\theta/2}\bigg(\frac{1}{2^m}\bigg)^{\frac{1}{2}h\theta-\frac{1}{8}\theta}\bigg], \tag{7}$$

where C is a generic constant and also depends on θ, $h(:=\frac{1}{q})$, $w_1(x',x'')$, *and c in* (1).

The proof of the proposition is based on (5) and the following estimate (8). To see (8), first by using Tanaka's formula (cf. [4] and [5]) and $\sum_{x^*_k\leqslant x}\hat{L}^{x^*_k}_t=\int_0^t 1_{\{X(s)\leqslant x\}}\,\mathrm{d}V_s=V_t-V_0-\int_0^t 1_{\{X(s)>x\}}\,\mathrm{d}V_s$, we have that

$$\tilde{L}^x_t=(X_t-x)^+-(X_0-x)^+-\int_0^t 1_{\{X_s>x\}}\,\mathrm{d}M_s-(V_t-V_0):=\varphi_t(x)-\int_0^t 1_{\{X_s>x\}}\,\mathrm{d}M_s-(V_t-V_0).$$

Now by using the following estimates in the proof of Lemma 2.1 in [1]: for any $\gamma\geqslant 1$ and $a_i<a_{i+1}$, $|\varphi_t(a_{i+1})-\varphi_t(a_i)|^\gamma\leqslant 2^\gamma(a_{i+1}-a_i)^\gamma$, $E|\int_0^t 1_{\{a_i<X_s\leqslant a_{i+1}\}}\,\mathrm{d}M_s|^\gamma\leqslant c(a_{i+1}-a_i)^{\gamma/2}$, we have that

$$\begin{aligned}E\big[\Delta^{m+1}_{2r-1}\tilde{L}^x_t\,\Delta^{m+1}_{2l-1}\tilde{L}^x_t\big]&\leqslant C\bigg[\bigg(\frac{1}{2^{m+1}}\bigg)^2 w_1(x',x'')^2+2\bigg(\frac{1}{2^{m+1}}\bigg)^{3/2}w_1(x',x'')^{3/2}\bigg]\\&\quad+E\bigg|\int_0^t 1_{\{x^{m+1}_{2r-2}<X_s\leqslant x^{m+1}_{2r-1}\}}1_{\{x^{m+1}_{2l-2}<X_s\leqslant x^{m+1}_{2l-1}\}}\,\mathrm{d}\langle M\rangle_s\bigg|\\&\leqslant\begin{cases}C\Big(\frac{1}{2^{m+1}}\Big)^{3/2}, & \text{if } r\neq l,\\ C\frac{1}{2^{m+1}}, & \text{if } r=l.\end{cases}\end{aligned}\tag{8}$$

Here C is a generic constant and also depends on $w_1(x',x'')$.

From the estimations of Propositions 2, 3 and Proposition 4.1.2 in [3], we obtain:

Theorem 4. *Assume* $2\leqslant q<4$. *Let* $q<\theta<4$. *Then for the continuous process* $Z_x=(\tilde{L}^x_t,g(x))$ *satisfying* (1), *there exists a unique* $\mathbf{Z}^i$ *on* Δ *taking values in* $(R^2)^{\otimes i}$ $(i=1,2)$ *such that*

$$\sum_{i=1}^2\sup_D\bigg(\sum_l\big|\mathbf{Z}(m)^i_{x_{l-1},x_l}-\mathbf{Z}^i_{x_{l-1},x_l}\big|^{\theta/i}\bigg)^{i/\theta}\to 0,$$

both almost surely and in $L^1(\Omega,\mathcal{F},P)$ *as* $m\to\infty$. *In particular, when* $2\leqslant q<3$, $\mathbf{Z}=(1,\mathbf{Z}^1,\mathbf{Z}^2)$ *is the canonical geometric rough path associated with* $Z_\cdot$. *Moreover,* $\mathbf{Z}^1_{a,b}=Z_b-Z_a$.

In the following, we will only consider the case that $2\leqslant q<3$ and take $\theta\in(q,3)$. From Chen's identity, it is easy to know that for any $(a,b)\in\Delta$, $\mathbf{Z}^2_{a,b}=\lim_{m(D_{[a,b]})\to 0}\sum_{i=0}^{r-1}(\mathbf{Z}^2_{x_i,x_{i+1}}+\mathbf{Z}^1_{a,x_i}\otimes\mathbf{Z}^1_{x_i,x_{i+1}})$. So there exists:

$$\begin{aligned}&\lim_{m(D_{[a,b]})\to 0}\sum_{i=0}^{r-1}\big((\mathbf{Z}^2_{x_i,x_{i+1}})_{2,1}+g(x_i)(\tilde{L}^{x_{i+1}}_t-\tilde{L}^{x_i}_t)\big)\\&=\lim_{m(D_{[a,b]})\to 0}\sum_{i=0}^{r-1}\big((\mathbf{Z}^2_{x_i,x_{i+1}})_{2,1}+\big(g(x_i)-g(a)\big)(\tilde{L}^{x_{i+1}}_t-\tilde{L}^{x_i}_t)\big)+g(a)(\tilde{L}^b_t-\tilde{L}^a_t)\\&=(\mathbf{Z}^2_{a,b})_{2,1}+g(a)(\tilde{L}^b_t-\tilde{L}^a_t).\end{aligned}\tag{9}$$

Denote this limit by $\int_a^b g(x)\,\mathrm{d}\tilde{L}_t^x$. Here $(\mathbf{Z}_{a,b}^2)_{2,1}$ is the lower-left element of the 2×2 matrix $\mathbf{Z}_{a,b}^2$.

Let $Z_j(x) := (\tilde{L}_t^x, g_j(x))$, where $g_j(\cdot)$ is of bounded q-variation uniformly in j for $2 \leqslant q < 3$, and when $j\to\infty$, $g_j(x)\to g(x)$ for all $x\in R$. Repeating the above argument, for each j, we can find the canonical geometric rough path $\mathbf{Z}_j = (1, \mathbf{Z}_j^1, \mathbf{Z}_j^2)$ associated with Z_j, the integral $\int_a^b g_j(x)\,\mathrm{d}\tilde{L}_t^x = ((\mathbf{Z}_j)_{a,b}^2)_{2,1} + g_j(a)(\tilde{L}_t^b - \tilde{L}_t^a)$, and the smooth rough path $\mathbf{Z}_j(m) = (1, \mathbf{Z}_j(m)^1, \mathbf{Z}_j(m)^2)$. Actually, $(\mathbf{Z}_j)_{a,b}^1 \to \mathbf{Z}_{a,b}^1$ in the sense of the uniform topology, and also in the sense of the θ-variation topology. As for $(\mathbf{Z}_j)_{a,b}^2$, we can easily see that

$$d_{2,\theta}\big((\mathbf{Z}_j)^2, \mathbf{Z}^2\big) \leqslant d_{2,\theta}\big((\mathbf{Z}_j)^2, \big(\mathbf{Z}_j(m)\big)^2\big) + d_{2,\theta}\big(\big(\mathbf{Z}_j(m)\big)^2, \mathbf{Z}(m)^2\big) + d_{2,\theta}\big(\mathbf{Z}(m)^2, \mathbf{Z}^2\big). \tag{10}$$

From Theorem 4, we know that $d_{2,\theta}(\mathbf{Z}(m)^2, \mathbf{Z}^2) \to 0$ as $m\to\infty$. Moreover, it is not difficult to see from the proofs of Propositions 2, 3, and Theorem 4, that $d_{2,\theta}((\mathbf{Z}_j)^2, (\mathbf{Z}_j(m))^2) \to 0$ as $m\to\infty$ uniformly in j. It is also easy to prove that, for any fixed m, $d_{2,\theta}((\mathbf{Z}_j(m))^2, \mathbf{Z}(m)^2) \to 0$ as $j\to\infty$. Hence it follows from a standard argument that $d_{2,\theta}((\mathbf{Z}_j)^2, \mathbf{Z}^2) \to 0$ as $j\to\infty$. This implies that $(\mathbf{Z}_j)_{a,b}^2 \to \mathbf{Z}_{a,b}^2$ as $j\to\infty$. Then by (9) and the definition of $\int_a^b g_j(x)\,\mathrm{d}\tilde{L}_t^x$, we know that $\int_a^b g_j(x)\,\mathrm{d}\tilde{L}_t^x \to \int_a^b g(x)\,\mathrm{d}\tilde{L}_t^x$ as $j\to\infty$. Note now that the local time L_t^x has a compact support in x a.s. So it is easy to see from taking $[x', x'']$ covering the support of L_t^x that the above construction of the integrals and the convergence can work for the integrals on R. Therefore we have:

Proposition 5. *Let $Z_j(x) := (\tilde{L}_t^x, g_j(x))$, $Z(x) := (\tilde{L}_t^x, g(x))$, where $g_j(\cdot)$, $g(\cdot)$ are of bounded q-variation uniformly in j, $2\leqslant q<3$. Assume $g_j(x)\to g(x)$ as $j\to\infty$ for all $x\in R$. Then as $j\to\infty$, $\mathbf{Z}_j(\cdot)\to\mathbf{Z}(\cdot)$ a.s. in the θ-variation distance. Moreover, as $j\to\infty$, $\int_{-\infty}^{\infty} g_j(x)\,\mathrm{d}\tilde{L}_t^x \to \int_{-\infty}^{\infty} g(x)\,\mathrm{d}\tilde{L}_t^x$ a.s.*

For the jump part $h(t,x)$ of the local time, from Lebesgue's dominated convergence theorem, $\int_{-\infty}^{\infty} g_j(x)\,\mathrm{d}h(t,x) \to \int_{-\infty}^{\infty} g(x)\,\mathrm{d}h(t,x)$, as $j\to\infty$. So we can get $\int_{-\infty}^{\infty} g_j(x)\,\mathrm{d}L_t^x \to \int_{-\infty}^{\infty} g(x)\,\mathrm{d}L_t^x$, as $j\to\infty$. Note that we can choose smooth g_j such that $g_j(x)\to g(x)$ as $j\to\infty$ for each x. In this case, the rough path integral $\int_{-\infty}^{\infty} g_j(x)\,\mathrm{d}\tilde{L}_t^x$ agrees with the Riemann integral and converges to the rough path integral $\int_{-\infty}^{\infty} g(x)\,\mathrm{d}\tilde{L}_t^x$. If $g(x)$ has finite discontinuities, we can treat it easily by considering the integral piece by piece. Finally, we deduce an extension of the Tanaka–Meyer formula by a smoothing procedure.

Theorem 6. *Let $X = (X_t)_{t\geqslant 0}$ be a continuous semimartingale and $f: R\to R$ be an absolutely continuous function and have left derivative $\nabla^- f$. Assume that, $\nabla^- f$ is left continuous with finite discontinuities, locally bounded, and is of bounded q-variation, where $1\leqslant q<3$. Then $\int_{-\infty}^{\infty} \nabla^- f(x)\,\mathrm{d}_x L_t^x$ has a modification which is continuous in t such that P-a.s.*

$$f(X_t) = f(X_0) + \int_0^t \nabla^- f(X_s)\,\mathrm{d}X_s - \int_{-\infty}^{\infty} \nabla^- f(x)\,\mathrm{d}_x L_t^x, \quad 0\leqslant t<\infty. \tag{11}$$

Here the integral $\int_{-\infty}^{\infty} \nabla^- f(x)\,\mathrm{d}_x L_t^x$ is a Lebesgue–Stieltjes integral when $q=1$, a Young integral when $1<q<2$, and a Lyons' rough path integral when $2\leqslant q<3$, respectively.

Remark 1. Although $\int_{-\infty}^{\infty} \nabla^- f(x)\,\mathrm{d}_x L_t^x$ has a continuous version, it is not clear whether or not $\mathbf{Z}(t)$ in the rough path space $(\Omega_\theta, d_\theta)$ has a version which is continuous in t in the metric d_θ.

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