Available online at www.sciencedirect.com
ScienceDirect
C. R. Acad. Sci. Paris, Ser. I 346 (2008) 585–588
COMPTES RENDUS MATHEMATIQUE
http://france.elsevier.com/direct/CRASS1/

Calculus of Variations

Monge–Ampère equations and Bellman functions: The dyadic maximal operator

Leonid Slavin [a,1], Alexander Stokolos [b,2], Vasily Vasyunin [c,3]

[a] *Department of Mathematics, University of Missouri, Columbia, MO 65211, USA*
[b] *Department of Mathematics, DePaul University, Chicago, IL 60614, USA*
[c] *St. Petersburg Department of the V. A. Steklov Mathematical Institute, Russian Academy of Sciences, Russia*

Received 26 February 2008; accepted 28 February 2008

Available online 11 April 2008

Presented by Jean-Pierre Kahane

Abstract

We find explicitly the Bellman function for the dyadic maximal operator on L^p as the solution of a Bellman partial differential equation of Monge–Ampère type. This function has been previously found by A. Melas (2005) in a different way, but it is our partial differential equation-based approach that is of principal interest here. Clear and replicable, it holds promise as a unifying template for past and current Bellman function investigations. ***To cite this article: L. Slavin et al., C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Équations de Monge–Ampère et fonctions de Bellman : l'opérateur maximal dyadique. Nous construisons explicitement la fonction de Bellman pour l'opérateur maximal dyadique sur L^p comme solution d'une équation aux dérivées partielles de Bellman de type Monge–Ampère. La fonction a été introduite par A. Melas (2005) sous un angle différent, mais ici nous privilégions notre approche à partir d'une équation aux dérivées partielles. Claire et reproductible, cette approche peut servir de principe unificateur dans les investigations passées et actuelles concernant les fonctions de Bellman. ***Pour citer cet article : L. Slavin et al., C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

E-mail addresses: leonid@math.missouri.edu (L. Slavin), astokolo@depaul.edu (A. Stokolos), vasyunin@pdmi.ras.ru (V. Vasyunin).

[1] L. Slavin's research supported in part by the NSF (grant DMS-0701254).

[2] A. Stokolos' research supported in part by a grant from the Faculty Research and Development Fund of the College of Liberal Arts and Sciences of DePaul University.

[3] V. Vasyunin's research supported in part by RFBR (grant 05-01-00925).

1631-073X/$ – see front matter
doi:10.1016/j.crma.2008.03.003

1. Introduction

For a locally integrable function g on $\mathbb{R}^n$ and a set $E \subset \mathbb{R}^n$ with $|E| \neq 0$, let $\langle g\rangle_E = \frac{1}{|E|}\int_E g$ be the average of g over E. Let $p > 1$ and $q > 1$ be conjugate exponents, i.e. $p^{-1} + q^{-1} = 1$. Let φ be a nonnegative locally L^p-function on $\mathbb{R}^n$. Fix a dyadic lattice D on $\mathbb{R}^n$ and consider the dyadic maximal operator:

$$M\varphi(x) = \sup_{I \ni x;\, I \in D} \langle \varphi\rangle_I.$$

Following F. Nazarov and S. Treil [2], we define the Bellman function for $M\varphi$,

$$\boldsymbol{B}(f, F, L) = \sup_{0 \leqslant \varphi \in L^p_{\mathrm{loc}}(\mathbb{R}^n)} \Bigl\{ \bigl\langle (M\varphi)^p \bigr\rangle_Q \colon \langle\varphi\rangle_Q = f;\ \bigl\langle \varphi^p \bigr\rangle_Q = F;\ \sup_{R \supset Q} \langle \varphi\rangle_R = L \Bigr\}. \tag{1}$$

Observe that $\boldsymbol{B}$ is independent of Q and well-defined on the domain: $\Omega = \{(f, F, L)\colon\ 0 < f \leqslant L;\ f^p \leqslant F\}$. Finding $\boldsymbol{B}$ will, among other things, provide a sharp refinement of the Hardy–Littlewood–Doob maximal inequality

$$\|M\varphi\|_p \leqslant q\|\varphi\|_p. \tag{2}$$

In [2], the authors show that $\boldsymbol{B}(f, F, L) \leqslant q^p F - pqfL^{p-1} + pL^p$, which implies (2). A. Melas in [1], using deep combinatorial properties of the operator M and without relying on the Bellman PDE, finds $\boldsymbol{B}$ explicitly. In contrast, we develop a boundary value problem of Monge–Ampère type that $\boldsymbol{B}$ must satisfy (assuming sufficient differentiability) and solve it, producing the function from [1]. Our approach has been used as the foundation of several recent Bellman function results. We first restrict our attention to the one-dimensional case and then show that the Bellman function does not depend on dimension.

2. Finite-differential and differential properties of $\boldsymbol{B}$

Let Q be an interval and Q_-, Q_+ its left and right halves, respectively. Let $(f_\pm, F_\pm) = (f_{Q_\pm}, F_{Q_\pm})$, $(f, F) = ((f_-, F_-) + (f_+, F_+))/2$. Taking suprema in the identity

$$\bigl\langle (M\varphi)^p \bigr\rangle_Q = \frac{1}{2}\bigl\langle (M\varphi)^p \bigr\rangle_{Q_-} + \frac{1}{2}\bigl\langle (M\varphi)^p \bigr\rangle_{Q_+},$$

over all φ with appropriate averages, we obtain:

$$\boldsymbol{B}(f, F, L) \geqslant \frac{1}{2}\boldsymbol{B}\bigl(f_-, F_-, \max\{f_-, L\}\bigr) + \frac{1}{2}\boldsymbol{B}\bigl(f_+, F_+, \max\{f_+, L\}\bigr). \tag{3}$$

Any function B satisfying this pseudo-concavity property on Ω will be a majorant of the true Bellman function. The following theorem phrases this condition in a differential form:

Theorem 2.1. *Let $z = (f, F)$. Assuming sufficient smoothness on the Bellman function B, condition* (3) *holds for all admissible triples if and only if*:

$$\det\biggl(\frac{\partial^2 B}{\partial z^2}\biggr) = 0, \quad B_{ff} \leqslant 0, \quad B_L \geqslant 0 \quad \text{on } \Omega; \qquad 2B_{fL} + B_{LL} \leqslant 0, \quad B_L = 0 \quad \text{when } f = L. \tag{4}$$

3. Homogeneity, boundary value problem, solution

We reduce the order of the PDE in (4) by using the multiplicative homogeneity of $\boldsymbol{B}$: $\boldsymbol{B}(f, F, L) = L^p\boldsymbol{B}(f/L, F/L^p, 1) \overset{\text{def}}{=} L^p G(x, y)$, where $x = f/L$, $y = F/L^p$. In addition, $F = f^p$ only for functions that are constant on Q, so $\boldsymbol{B}(f, f^p, L) = L^p$, meaning $G(x, x^p) = 1$. Coupling this with the first and the last conditions in (4), we get a boundary value problem for G on the domain $\{(x, y) \mid 0 < x \leqslant 1;\ x^p \leqslant y\}$:

$$G_{xx}G_{yy} = G_{xy}^2; \quad G(x, x^p) = 1; \quad pG(1, y) = G_x(1, y) + pyG_y(1, y). \tag{5}$$

We look for the solution of the Monge–Ampère equation (5) in the general parametric form:

$$G(x, y) = tx + f(t)y + g(t); \qquad x + f'(t)y + g'(t) = 0. \tag{6}$$

Fix a value of t, i.e. fix one of the straight-line trajectories in (6). Let $(u(t), u^p(t))$ be the point where that trajectory intersects the lower boundary $y = x^p$. We have:

$$G(u, u^p) = tu(t) + f(t)u^p(t) + g(t) = 1; \qquad u(t) + f'(t)u^p(t) + g'(t) = 0.$$

Differentiating the first equation and using the second one, we get, after some algebra, $f = -t/(pu^{p-1})$, $g = 1 - tu/q$. Assume now that the trajectory intersects the right boundary $x = 1$ at the point $(1, v(t))$. Then $G(1, v) = t + fv + g$. On the other hand, parametrization (6) implies $G_x = t$, $G_y = f(t)$ and so the second boundary condition in (5) becomes $G(1, v) = \frac{t}{p} + fv$. This gives $g = -t/q$, allowing us to express $t = q/(u - 1)$. Simplifying, we obtain a complete solution of the form (6):

$$G(x, y) = \frac{y}{u^p}; \qquad x - \frac{qu - 1}{qu^p} y - \frac{1}{q} = 0. \tag{7}$$

In terms of the original variables, we get a Bellman function candidate near the boundary $f = L$:

$$B(f, F, L) = Fu^{-p}(f/L, F/L^p). \tag{8}$$

4. From the candidate to the true function

4.1. Condition $B \geqslant \boldsymbol{B}$

One can readily verify that the rest of conditions (4) are satisfied by the candidate (8). Therefore, property (3) holds and one can perform the Bellman induction: take any nonnegative function $\varphi \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ and an interval $Q_0 \in D$. For an interval $Q \subset Q_0$, $Q \in D$, let $X_Q = (f_Q, F_Q, L_Q)$ with f, F, and L defined as in (1). Then

$$\begin{aligned} B(f_{Q_0}, F_{Q_0}, L_{Q_0}) &\geqslant \frac{1}{2}B(X_{(Q_0)_-}) + \frac{1}{2}B(X_{(Q_0)_+}) \\ &\geqslant \frac{1}{|Q_0|} \sum_{Q\subset Q_0,\ |Q|=2^{-n}|Q_0|} |Q| B(X_Q) \geqslant \frac{1}{|Q_0|} \sum_{Q\subset Q_0,\ |Q|=2^{-n}|Q_0|} |Q| L_Q^p \\ &= \frac{1}{|Q_0|} \sum_{Q\subset Q_0,\ |Q|=2^{-n}|Q_0|} |Q| \Big(\sup_{R\supset Q} \langle \varphi \rangle_R \Big)^p \longrightarrow \big\langle (M\varphi)^p \big\rangle_{Q_0}, \quad \text{as } n \to \infty. \end{aligned} \tag{9}$$

Here we have used that $B \geqslant L^p$. Taking supremum on the right over all φ with the above X_{Q_0} we get $B \geqslant \boldsymbol{B}$.

4.2. Condition $B \leqslant \boldsymbol{B}$

To get the reverse inequality, we need to construct, for every point $(f, F, L) \in \Omega$, a sequence of nonnegative functions on $(0, 1)$, $\{\varphi_n\}$, so that

$$\lim_{n\to\infty} \big\langle (M\varphi_n)^p \big\rangle_{(0,1)} \geqslant B(f, F, L).$$

To do this, we use the trajectories $t = \text{const}$ of the Monge–Ampère equation from Section 3. In the original variables, this gives:

$$f = \frac{L}{q} + AF. \tag{10}$$

On the boundary $f = L$ going along these trajectories yields the extremal sequence

$$\varphi_n(t) = \begin{cases} \alpha_n L, & 0 < t < 2^{-n}, \\ \varphi_n(2^k t - 1), & 2^{-k} < t < 2^{-k+1}, k = 2, \ldots, n, \\ \beta_n \varphi_n(2t - 1), & \frac{1}{2} < t < 1. \end{cases} \tag{11}$$

The definition is understood recursively, whereby the function is defined on a portion of $(0, 1)$, then on the same portion of the remaining part, and so on. The numbers α_n and β_n are chosen so that $\langle \varphi_n \rangle_{(0,1)} = L$ and $\langle \varphi_n^p \rangle_{(0,1)} = F$. This means

$$\frac{1}{2^n}\alpha_n + \frac{1}{2}\beta_n = \frac{1}{2^n} + \frac{1}{2}; \qquad \frac{1}{2^n}\alpha_n^p + \frac{1}{2}\beta_n^p \frac{F}{L^p} = \left(\frac{1}{2^n} + \frac{1}{2} \right) \frac{F}{L^p}.$$

One can show that $\alpha_n M\varphi_n \geqslant \varphi_n$ and $\alpha_n \to u(1, F/L^p)$ with u defined by (7). Therefore,

$$\lim_{n\to\infty}\big\langle (M\varphi_n)^p\big\rangle_{(0,1)} \geqslant \lim_{n\to\infty}\frac{1}{\alpha_n^p}\big\langle \varphi_n^p\big\rangle_{(0,1)} = \lim_{n\to\infty}\frac{F}{\alpha_n^p} = Fu^{-p}(1, F/L^p) = \boldsymbol{B}(L, F, L),$$

which gives $\boldsymbol{B}(L, F, L) \geqslant B(L, F, L)$.

On the boundary $F = f^p$ the situation is simple: here the only test functions are constants and so $B(f, f^p, L) = \boldsymbol{B}(f, f^p, L) = L^p$. Having constructed the extremal sequences on the two boundaries, we get the extremal sequence at any point (f, F, L) with $f > L/q$ as their weighted dyadic rearrangement built along the unique extremal trajectory of the form (10) passing through the point.

One observes, however, that trajectories (10) cannot be used with $A < 0$, since they then would intersect the "forbidden" boundary $f = 0$. (It is forbidden because, for a nonnegative function, $f = 0$ implies $F = 0$.) In fact, in the region $0 < f < L/q$, no trajectory can lean either to the left or to the right (the forbidden boundary to the left, the existing extremal trajectory $f = L/q$ to the right). We conclude two things: the trajectories are vertical in this region and the candidate (8) no longer works there. However, this is quickly rectified: If $G(x, y) = a(x)y + b(x)$, then $G(x, x^p) = 1$ implies that $G(x, y) = 1 + a(x)(y - x^p)$. Now $G_{xx}G_{yy} - G_{xy}^2 = -(a'(x))^2 = 0$, and $G(1/q, y) = q^p y$ implies that $a(x) = q^p$. Thus we get the unique two-piece Bellman function candidate:

$$B(f, F, L) = \begin{cases} Fu^{-p}(f/L, F/L^p), & L < qf, \\ L^p + q^p(F - f^p), & L \geqslant qf. \end{cases} \tag{12}$$

(In the notation of [1], $u^{-p}(x, y) = \omega_p((px - p + 1)/y)^p$.) This B still satisfies (3). Therefore, Bellman induction (9) works. We now need an extremal sequence proving that $\boldsymbol{B} \geqslant B$ in the region $L \geqslant qf$. There is a unique extremal trajectory passing through each point of the region. However, the trajectory is vertical and so intersects the boundary of Ω at a single point; as a result we cannot use a weighted average of boundary extremal sequences like we just did for the region $L > f/q$. We deal with it by tilting the trajectory slightly to the right, which produces a (distant) second boundary point, at the boundary $f = L$. This lets us use the extremal sequence φ_n from (11), while simultaneously reducing the tilt. Namely, fix (f, F, L) and $k \geqslant 1$. Define γ_k and F_k so that $L - \gamma_k = 2^k(f - \gamma_k)$ and $F_k - \gamma_k^p = 2^k(F - \gamma_k^p)$. (Observe that $\gamma_k \to f$ and $F_k \to \infty$.) Using (11), form a sequence $\{\varphi_{k,n}\}_{n=1}^{\infty}$ with $\langle \varphi_{k,n}\rangle_{(0,1)} = L$ and $\langle \varphi_{k,n}^p\rangle_{(0,1)} = F_k$, so that $\langle (M\varphi_{k,n})^p\rangle_{(0,1)} \to B(L, F_k, L)$, as $n \to \infty$. Let,

$$\psi_{k,n}(t) = \begin{cases} \varphi_{k,n}(2^k t), & 0 < t < 2^{-k}, \\ \gamma_k, & 2^{-k} < t < 1, \\ 2L - f, & 1 < t < 2. \end{cases}$$

Direct computation shows that $\langle \psi_{k,n}\rangle_{(0,1)} = f$, $\langle \psi_{k,n}^p\rangle_{(0,1)} = F$, and $\langle \psi_{k,n}\rangle_{(0,2)} = L$. Then

$$\begin{aligned}\big\langle (M\psi_{k,n})^p\big\rangle_{(0,1)} &\geqslant L^p(1 - 2^{-k}) + 2^{-k}\big\langle (M\varphi_{k,n})^p\big\rangle_{(0,1)} \xrightarrow[n\to\infty]{} L^p(1 - 2^{-k}) + 2^{-k}B(L, F_k, L) \\ &\xrightarrow[k\to\infty]{} L^p + (F - f^p)u^{-p}(1, \infty) = L^p + q^p(F - f^p).\end{aligned}$$

5. Several dimensions

It turns out that the Bellman function (1), (12) is dimension-free. Fix a dyadic cube Q and let $Q_1, \ldots, Q_{2^n}$ be its dyadic offspring. Then

$$B\left(2^{-n}\sum_{k=1}^{2^n} z_k, L\right) \geqslant 2^{-n}\sum_{k=1}^{n} B\big(z_k, \max\{f_k, L\}\big).$$

Therefore, we can run the induction (9) to prove that $B \geqslant \boldsymbol{B}$. The other direction is shown by a trivial modification of the one-dimensional maximizing sequences. A similar argument can be used to show that the same Bellman function works for the maximal operator on trees, the setting of choice in [1].

References

[1] A. Melas, The Bellman functions of dyadic-like maximal operators and related inequalities, Adv. Math. 192 (2) (2005) 310–340.

[2] F. Nazarov, S. Treil, The hunt for Bellman function: applications to estimates of singular integral operators and to other classical problems in harmonic analysis, Algebra i Analiz 8 (5) (1996) 32–162 (in Russian). Translation in St. Petersburg Math. J. 8 (5) (1997) 721–824.