Global weak solutions for asymmetric incompressible fluids with variable density

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Abstract

We establish the existence of global in time weak solutions for the equations of asymmetric incompressible fluids with variable density, when the initial density is not necessarily strictly positive. To cite this article: P. Braz e Silva, E.G. Santos, C. R. Acad. Sci. Paris, Ser. I 346 (2008).

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Résumé


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1. Introduction

Let \( \Omega \subset \mathbb{R}^3 \) be an open bounded set. We are interested in the flow of an asymmetric incompressible fluid with variable density in \( \Omega \). So, for a given time \( T > 0 \), we consider the equations

\[
\begin{align*}
\rho (u_t + (u \cdot \nabla) u) - (\mu + \mu_r) \Delta u + \nabla p &= 2\mu_r \text{curl } w + \rho f, \\
\rho (w_t + (u \cdot \nabla) w) - (c_0 + c_d) \Delta w - (c_0 + c_d - c_a) \nabla (\text{div } w) + 4\mu_r w &= 2\mu_r \text{curl } u + \rho g, \\
\text{div } u &= 0, \\
\rho_f + u \cdot \nabla \rho &= 0,
\end{align*}
\]

in \( \Omega \times (0, T) \), with initial and boundary conditions

\[
\begin{align*}
u(x, t) &= w(x, t) = 0, & \forall (x, t) \in \partial \Omega \times (0, T), \\
u(x, 0) &= u_0(x), & w(x, 0) = w_0(x), & \rho(x, 0) = \rho_0(x), & \forall x \in \Omega.
\end{align*}
\]
The unknowns $u$, $w$, $\rho$, and $\rho$ are, respectively, the linear velocity, the angular velocity of rotation of fluid particles, the mass density and the pressure distribution of the fluid. The functions $f$ and $g$ are given external forces. The positive constants $\mu$, $\mu_r$, $c_0$, $c_g$, $c_d$ are related with viscosity properties of the fluid, and satisfy $c_0 + c_d > c_d$.

For the derivation of Eqs. (1) and a discussion about their physical meaning, see [3]. Concerning applications, the micropolar fluid model has been used, for example, in lubrication theory [4,7], as well as in modelling blood flow in thin vessels [1].

In [2], some existence and uniqueness results for strong solutions are given, in the case of a strictly positive initial density. In [5], local in time existence of weak solutions was established (see also [6]). For this local result though, the initial density is required to satisfy the integrability condition $\|\rho_0^{-1}\|_{L^1} < \infty$. Here, we announce the existence of global in time weak solutions, requiring the initial density to be only nonnegative, that is, $\rho_0 \geq 0$ (see Theorem 2.1).

Our results bring the knowledge about weak solutions of system (1) to the same level of the knowledge about weak solutions of the variable density Navier–Stokes system [8,9].

2. Preliminaries

We denote by $D(\Omega)$ the space of test functions defined in $\Omega$, and by $D'(\Omega)$ the space of distributions over $\Omega$. We use the usual notation for Sobolev spaces

$$W^{m,q}(\Omega) = \{ f \in L^q(\Omega); \| D^\alpha f \|_{L^q(\Omega)} < +\infty, |\alpha| \leq m \},$$

for a multi-index $\alpha$, a nonnegative integer $m$ and $1 \leq q \leq +\infty$. We write $H^m(\Omega) := W^{m,2}(\Omega)$ and denote by $H^m_0(\Omega)$ the closure of $D(\Omega)$ in $H^m(\Omega)$. If $B$ is a Banach space and $T > 0$, we denote by $L^q([0, T]; B)$ the Banach space of $B$ valued functions defined on the interval $[0, T]$ that are $L^q$-integrable in Bochner’s sense. By $D(0, T; B)$, we denote the space of $B$ valued $C^\infty$ functions defined on $[0, T]$, with compact support in $(0, T)$. Accordingly, we indicate the space of distributions with values in $B$ by $D'(0, T; B)$. As it is usual in this context, we denote $\mathbb{R}^3$ valued functions by bold face letters. We write

$$V = \{ v \in (D(\Omega))^3; \nabla \cdot v = 0 \},$$

and denote by $H$ and $V$ the closure of $V$ in $(L^2(\Omega))^3$ and $(H^1_0(\Omega))^3$ respectively.

Some spaces which are not so standard but play an important role in our results are the Nikolskii spaces, defined as follows: Let $B$ be a Banach space. Given a function $f : (0, T) \rightarrow B$ and $h > 0$, let $\tau_h f : (-h, T - h) \rightarrow B$ be the translated function of $f$, defined by $(\tau_h f)(t) = f(t + h)$. For $1 \leq q \leq \infty$, $0 < \varsigma < 1$, the Nikolskii space $N^{0,q}$ is defined by

$$N^{\varsigma,q}(0, T; B) := \{ f \in L^q(0, T; B); \sup_{h > 0} h^{-\varsigma} \| \tau_h f - f \|_{L^q(0, T; B)} < \infty \}.$$ 

The space $N^{\varsigma,q}(0, T; B)$ is a Banach space with respect to the norm

$$\| f \|_{N^{\varsigma,q}(0, T; B)} := \| f \|_{L^q(0, T; B)} + \sup_{0 < h < T} \left[ h^{-\varsigma} \| \tau_h f - f \|_{L^q(0, T; B)} \right].$$

One may see [9–11] for compactness properties of Nikolskii spaces. Our result is the following:

**Theorem 2.1.** Let $\Omega \subset \mathbb{R}^3$ be an open bounded set with Lipschitz boundary. Given $T > 0$, if $u_0 \in H$, $w_0 \in (L^2(\Omega))^3$, $\rho_0 \in L^\infty(\Omega)$, $\rho_0 \geq 0$, and $f, g \in L^1(0, T; (L^2(\Omega))^3)$, then there exist

$$u \in L^2(0, T; V), \quad w \in L^2(0, T; (H^1_0(\Omega))^3), \quad p \in W^{-1,\infty}(0, T; L^2(\Omega)), \quad \rho \in L^\infty(0, T; L^\infty(\Omega)),$$

such that

$$\rho u, \quad \rho w \in L^\infty(0, T; (L^2(\Omega))^3) \cap N^{\frac{1}{2},2}(0, T; (W^{-1,3}(\Omega))^3),$$

$$\inf_{\Omega} \rho(x, t) \leq \sup_{\Omega} \rho_0,$$

satisfying the equations

[Note: The equations are not explicitly written in the text provided.]
\[
\frac{\partial \rho u}{\partial t} + \text{div}(\rho u u) - (\mu + \mu_r) \Delta u + \nabla p = 2\mu_r \text{curl} w + \rho f,
\]
\[
\frac{\partial \rho w}{\partial t} + \text{div}(\rho u w) - (c_a + c_d) \Delta w - (c_0 + c_d - c_a) \nabla \text{div} w + 4\mu_r w = 2\mu_r \text{curl} u + \rho g,
\]
\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0, \quad \text{div} u = 0,
\]
in \(\Omega \times (0, T)\), the boundary conditions (2), and the weak initial conditions
\[
\rho|_{t=0} = \rho_0, \quad \left( \int_{\Omega} \rho u \cdot v \, dx \right)(0) = \int_{\Omega} \rho_0 u_0 \cdot v, \quad \forall v \in V,
\]
\[
\left( \int_{\Omega} \rho w \cdot z \, dx \right)(0) = \int_{\Omega} \rho_0 w_0 \cdot z, \quad \forall z \in \left(H^1_0(\Omega)\right)^3.
\]

**Remark 1.** In [9], the author obtains \(\rho u \in N^{1.2}_0(0, T; (W^{-1.2}(\Omega))^3)\). The theorem above assures improved regularity for \(\rho u, \rho w\), and its counterpart for the variable density Navier–Stokes was actually established in [8].

**Sketch of the proof.** The proof of Theorem 2.1 is based on a semi-Galerkin method, adapting the techniques used in [9] for the variable density Navier–Stokes system to treat our case of asymmetric fluids. First, one defines \(V^m, W^m\), suitable finite dimensional subspaces of \(V\) and \(L^2(\Omega)^3\), respectively. Then, choose sequences \(f^m, g^m \in C(([0, T]; (L^2(\Omega))^3), u^m_0 \in V^m, w^m_0 \in W^m, and \rho^m_0 \in C^1(\Omega)\) such that
\[
1/m + \inf_{\Omega} \rho_0 \leq \rho^m_0 \leq 1/m + \sup_{\Omega} \rho_0 \quad \text{in} \ \Omega, \quad \text{for all} \ m = 1, 2, \ldots,
\]
and \(f^m \to f, g^m \to g\) in \(L^1(0, T; (L^2(\Omega))^3)\), \(u^m_0 \to u_0\) in \(H, w^m_0 \to w_0\) in \((L^2(\Omega))^3\), \(\rho^m_0 \to \rho_0\) weak-* in \(L^\infty(\Omega)\). Now, for \(m \in \mathbb{N}\), we call the triplet \((\rho^m, u^m, w^m)\) an \((m)\)th approximate solution of problem (1)–(3) if \(\rho^m \in C^1(\Omega)\), \(u^m \in C^1([0, T]; V^m)\), \(w^m \in C^1([0, T]; W^m)\) satisfy the equations
\[
\int_{\Omega} \left( \rho^m (u^m_t + (u^m \cdot \nabla) u^m - f^m) \cdot v + (\mu + \mu_r) \nabla u^m \cdot \nabla v - 2\mu_r w^m \cdot \text{curl} v \right) \, dx = 0, \quad \forall v \in V^m,
\]
\[
\int_{\Omega} \left( \rho^m (w^m_t + (u^m \cdot \nabla) w^m - g^m) \cdot z + (c_a + c_d) \nabla w^m \cdot \nabla z + (c_0 + c_d - c_a) \text{div} w^m \cdot \text{div} z + 4\mu_r w^m \cdot z - 2\mu_r u^m \cdot \text{curl} z \right) \, dx = 0, \quad \forall z \in W^m,
\]
\[
\rho^m t^m + u^m \cdot \nabla \rho^m = 0,
\]
and the initial conditions \(u^m|_{t=0} = u^m_0, w^m|_{t=0} = w^m_0\), and \(\rho^m|_{t=0} = \rho^m_0\) in \(\Omega\). After establishing the existence of approximate solutions, one derives a priori bounds for them. Since one does not have a positive lower bound for the initial density, this task is a little bit harder than usual. The key idea here is to derive bounds for the products \(\rho u, \rho w\).

In special, after considerable work one obtains
\[
\| \tau_h (\rho^m u^m) - \rho^m u^m \|_{L^2(0, T-h; (W^{-1.2}(\Omega))^3)} \leq Ch^{\frac{1}{2}},
\]
\[
\| \tau_h (\rho^m w^m) - \rho^m w^m \|_{L^2(0, T-h; (W^{-1.2}(\Omega))^3)} \leq Ch^{\frac{1}{2}},
\]
where \(C\) is independent of \(m\), which imply \(\rho u, \rho w \in N^{1.2}_0(0, T; (W^{-1.2}(\Omega))^3)\). These bounds, together with some other ones, allow one to pass to the limit \(m \to \infty\) obtaining the desired solution. \(\square\)
References