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Numerical Analysis

Small viscosity solution of linear scalar 1-D conservation laws with one discontinuity of the coefficient

Bruno Fornet [a,b]

[a] LATP, Université de Provence, 39, rue Joliot-Curie, 13453 Marseille cedex 13, France
[b] LMRS, Université de Rouen, avenue de l'Université, BP 12, 76801 Saint Étienne du Rouvray, France

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Abstract

While linear conservations laws have a classical well-defined solution for sufficiently regular coefficients, it is not the case when the coefficients are, for instance, discontinuous across a fixed hypersurface. In this case, another approach has to be proposed in order to answer the double concern of existence and uniqueness of a solution to the problem. We will focus mainly on showing such concerns can be solved by means of a small viscosity approach in 1-D scalar frameworks, in particular for expansive discontinuities of the coefficient. The obtained small viscosity solution is also the solution in the sense Bouchut and James or LeFloch for scalar equations. ***To cite this article: B. Fornet, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Solution à petite viscosité de lois de conservations linéaires scalaires 1-D avec un coefficient discontinu. Pour des coefficients suffisamment réguliers, les lois de conservations linéaires ont un sens classique bien établi. Cela cesse cependant d'être le cas lorsque les coefficients sont par exemple discontinus au travers d'une hypersurface fixée. Dans ce cas de figure, une autre approche doit être proposée pour répondre à la double préoccupation de l'existence et de l'unicité d'une solution au problème. Notre but va être principalement de montrer que, dans des cas scalaires 1-D, une approche à viscosité évanescente permet de répondre à ces préoccupations, en particulier dans le cas d'une discontinuité expansive du coefficient. ***Pour citer cet article : B. Fornet, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Version française abrégée

Dans cette Note, nous nous focalisons sur un problème hyperbolique scalaire écrit sous forme conservative en une dimension d'espace. Ce problème a une discontinuité du coefficient a localisée sur une interface d'équation $x = 0$, ce qui fait qu'il n'a pas de solution classique ou, en tout cas, pas de solution classique unique. Selon le signe du coefficient dans un voisinage de l'interface, supposée non-caractéristique, nous donnons la solution à petite viscosité du problème dans les différents cas de figure possibles. Nous esquissons la preuve du résultat dans le cas expansif qui

E-mail address: fornet@cmi.univ-mrs.fr.

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est tout particulièrement intéressant. Ensuite nous parlons d'autres résultats obtenus, toujours par le biais d'approches à petite viscosité, dans le cas de systèmes. Le comportement de la solution à petite viscosité de l'analogue non-conservatif du cas expansif développé ici est très intéressant mais nous ne le développerons pas dans cette note. En particulier, une perte de régularité de la solution à petite viscosité par rapport aux données apparaît dès que plusieurs dimensions d'espaces sont présentes [6].

1. Introduction and notations

In this note, we are interested in the small viscosity solution of the following hyperbolic problem with discontinuous coefficients $P(L_c, f, h)$:

$$P(L_c, f, h):\quad \begin{cases} L_c u = f(t,x), & (t,x) \in (0,T)\times\mathbb{R}, \\ u|_{t=0} = h, \end{cases}$$

where the operator L_c is of the form $\partial_t + \partial_x(a(t,x).)$, with the restriction of the coefficient a to the half spaces $\{x > 0\}$ and $\{x < 0\}$ belonging to the set of infinitely derivable functions bounded as well as their derivatives. Without imposing any corner compatibility assumptions, the functions f and h are simply assumed to belong to the set of infinitely derivable functions with compact support. We introduce the viscous regularization of the operator: $L_c^\varepsilon := L_c - \varepsilon\partial_x^2$. Although $P(L, f, h)$ has no obvious classical sense, for fixed positive ε, the problem $P(L_c^\varepsilon, f, h)$ has a unique classical solution which is continuous over $(0,T)\times\mathbb{R}$. This paper is devoted to the investigation of the small viscosity solution of $P(L, f, h)$ obtained as $\lim_{\varepsilon\to 0^+} u^\varepsilon$. Such an approach has been used by the authors in [8] in the neighboring framework of shockwaves study. Sueur worked on the neighboring question of the approximation of discontinuous solution by viscous approaches in [12]. We recover the notion of solution proposed by LeFloch [9] and Bouchut, James [1,2] when the function f is equal to zero. We insist, as for instance did LeFloch in [9], about the link between $P(L_c, f, h)$ and the nonconservative problem $P(L_{nc}, F, H)$ where $L_{nc} = \partial_t + a(t,x)\partial_x$: if $f = \partial_x F$ and $g = \partial_x G$ then u is obtained by derivation with respect to x of the solution v of $P(L_{nc}, F, H)$. The same remark holds for the viscously perturbed equations, denoting by $L_{nc}^\varepsilon := L_{nc} - \varepsilon\partial_x^2$, u^ε can be obtained as the derivative with respect to x of $P(L_{nc}^\varepsilon, F, H)$, v^ε which belongs to $C^1((0,T)\times\mathbb{R})$. We refer to the work of Dal Maso, LeFloch and Murat [3] on nonconservative products, which clarifies the sense to give to $P(L_{nc}, F, H)$ hence inducing a sense to give to the "dual" problem $P(L_c, f, h)$.

For fixed positive ε, $P(L_c^\varepsilon, f, h)$ can be viewed as a transmission problem: the equation is satisfied on the half-spaces $\{\pm x > 0\}$ and the quantities u^ε and $au^\varepsilon - \varepsilon\partial_x u^\varepsilon$ are conserved through the interface $\{x = 0\}$. Assuming that the coefficient does not vanish in a neighborhood of $\{x = 0\}$, three cases arise naturally for the problem at hand depending on the behavior of the interface. Theorem 1.1 immediately below states the small viscosity solution selected by our approach in each case:

Theorem 1.1.

(i) *For a compressive interface, which means that* $\operatorname{sign} a(t,x) = -\operatorname{sign}(x)$ *in a neighborhood of* $\{x = 0\}$, u^ε *converges in the sense of distributions towards a measure-valued solution* u *of the form* $u_L\mathbf{1}_{x<0} + u_R\mathbf{1}_{x>0} + C(t)\delta_{x=0}$, *where both* u_L *and* u_R *are in* L^2, C *is a known continuous function of* t *and* δ *stands for the Dirac measure. This recovers the result of Poupaud and Rascle* [11] *when* $f = 0$.

(ii) *For a traversing interface, which means that* $\operatorname{sign} a|_{x=0^+} = \operatorname{sign} a|_{x=0^-}$, u^ε *converges in* L^2 *towards the solution of the transmission problem satisfying the equation on each side of the boundary and the Rankine–Hugoniot condition stating that the flux* $a\cdot u$ *is conserved through* $\{x = 0\}$.

(iii) *For an expansive interface, which means that* $\operatorname{sign} a(t,x) = \operatorname{sign}(x)$ *in a neighborhood of* $\{x = 0\}$, u^ε *converges in* L^2 *towards the solution of the transmission problem satisfying the equation on each side of the boundary and two transmission conditions sating that both the solution* u *and the flux* $a\cdot u$ *are conserved through* $\{x = 0\}$.

This paper is mainly devoted to the proof of Theorem 1.2 which is a sharpened version of point (iii) in Theorem 1.1. Note that this point is the most troublesome to treat as far as uniqueness of the solution is concerned. In what follows, R subscripts [resp. L subscripts] will be used for denoting restrictions to the domain $\{x > 0\}$ [resp. $\{x < 0\}$]:

Theorem 1.2. *Let $\underline{u}$ be defined as the solution of the transmission problem*:

$$\begin{cases} \partial_t \underline{u}_R + \partial_x(a_R \underline{u}_R) = f_R, & \{x>0\}, \\ \partial_t \underline{u}_L + \partial_x(a_L \underline{u}_L) = f_L, & \{x<0\}, \\ \underline{u}_R|_{x=0} = \underline{u}_L|_{x=0} = 0, & \forall t \in (0,T), \\ \underline{u}_R|_{t=0} = h_R, u_L|_{t=0} = h_L. \end{cases} \tag{1}$$

There is $C>0$ such that, for all $0<\varepsilon<1$, there holds:

$$\|u^\varepsilon - \underline{u}\|_{L^\infty([0,T]:L^2(\mathbb{R}))} \leqslant C\varepsilon^{1/4}.$$

2. Sketch of proof of Theorem 1.2

Let Ω_R be $(0,T)\times\mathbb{R}^{*+}$. Consider now the vector field defined through: $(t,x)\mapsto \partial_t + a_R(t,x)\partial_x$. We will denote by Γ_R the characteristic curve passing through $t=0, x=0$ and tangent to this vector field. A parametrization of Γ_R is given by: $\Gamma_R = \{(t, x_R(t)), t\in(0,T)\}$, where x_R is the solution of the equation:

$$\begin{cases} \dfrac{\mathrm{d}x_R}{\mathrm{d}t}(t) = a_R\big(t, x_R(t)\big), & t\in(0,T), \\ x_R(0) = 0. \end{cases}$$

Let us denote by $\tilde{a}_R$ an arbitrary smooth extension of a_R to $\{x<0\}$. We define then φ_R as the solution of:

$$\begin{cases} \big(\partial_t + \tilde{a}_R(t,x)\partial_x\big)\varphi_R = 0, & (t,x)\in(0,T)\times\mathbb{R}, \\ \varphi_R|_{t=0} = x. \end{cases}$$

The obtained φ_R is in $C^\infty((0,T)\times\mathbb{R})$. Moreover, we have: $\Gamma_R = \{(t,x)\in\Omega_R\colon \varphi_R(t,x)=0\}$. Ω_L, Γ_L and φ_L are defined in a symmetric way and there holds: $\Gamma_L = \{(t,x)\in\Omega_L\colon \varphi_L(t,x)=0\}$. Note well that, by construction of φ_L and φ_R, we have:

Lemma 1. *There is c such that, for all $(t,x)\in\Gamma_R$, there holds*:

$$\big|\partial_x\varphi_R(t,x)\big| \geqslant c > 0, \qquad \big|\partial_x\varphi_L(t,x)\big| \geqslant c > 0.$$

We note for instance:

$$\Omega_L^+ = \big\{(t,x)\in\Omega_L\colon \varphi_L(t,x) > 0\big\},$$

where the "L" stands for "on left-hand side of Γ_L" and the $+$ is related to the sign of $\varphi_L(t,x)$. We define in the same manner: Ω_L^-, Ω_R^+ and Ω_R^-.

We will begin by constructing an approximate solution of the viscous problem viewed as a transmission problem. We perform the construction of an approximate solution separately on the four domains Ω_L^-, Ω_L^+, Ω_R^+ and Ω_R^-. We will denote by $u^\varepsilon_{\mathrm{app},L,+}$ the restriction of $u^\varepsilon_{\mathrm{app}}$ to Ω_L^+ and so on. Let us present the different profiles and their ansatz:

$$u^\varepsilon_{\mathrm{app},L,+}(t,x) = \sum_{n=0}^{M}\left(\underline{\mathbf{U}}_{L,n,+}(t,x) + \mathbf{U}^c_{L,n,+}\left(t, \frac{\varphi_L(t,x)}{\sqrt{\varepsilon}}\right)\right)\varepsilon^{n/2},$$

where the profiles $\underline{\mathbf{U}}_{n,L,+}$ belong to $H^\infty(\Omega_L^+)$ and the characteristic boundary layer profiles $\mathbf{U}^c_{n,L,+}(t,x,\theta_L)$ belong to $\mathrm{e}^{-\delta|\theta_L|}H^\infty((0,T)\times\mathbb{R}^{*+})$, for some $\delta>0$. We will take a similar ansatz for $u^\varepsilon_{\mathrm{app},L,-}$, $u^\varepsilon_{\mathrm{app},R,-}$ and $u^\varepsilon_{\mathrm{app},R,+}$ over their respective domains. We begin by constructing the underlined profiles $\underline{\mathbf{U}}_n$ by induction, the boundary layer profiles $\mathbf{U}^c_n$ are then computed as a last step. A good reference about the computation of characteristic boundary layer profiles is [7] by Guès. We construct our profiles such that, for all fixed $\varepsilon>0$, $u^\varepsilon_{\mathrm{app}}$ belongs to $C^1([0,T]\times\mathbb{R})$. In what follows, we will note:

$$\underline{\mathbf{U}}_{R,j}(t,x) := \underline{\mathbf{U}}_{R,j,+}(t,x)\mathbf{1}_{(t,x)\in\Omega_R^+} + \underline{\mathbf{U}}_{R,j,-}(t,x)\mathbf{1}_{(t,x)\in\Omega_R^-},$$

and:

$$\mathbf{U}^c_{R,j}\left(t,x,\frac{\varphi_R(t,x)}{\sqrt{\varepsilon}}\right) := \mathbf{U}^c_{R,j,+}\left(t,\frac{\varphi_R(t,x)}{\sqrt{\varepsilon}}\right)\mathbf{1}_{(t,x)\in\Omega_R^+} + \mathbf{U}^c_{R,j,-}\left(t,\frac{\varphi_R(t,x)}{\sqrt{\varepsilon}}\right)\mathbf{1}_{(t,x)\in\Omega_R^-}.$$

Actually, $\mathbf{U}^c_{R,j}$ is piecewise constant with respect to x on each side of Γ_R, which explains that $\mathbf{U}^c_{n,L,+}$ and $\mathbf{U}^c_{n,L,-}$ have no direct dependency in x. Of course, there holds: $\underline{u}_R := \underline{\mathbf{U}}_{R,0}$ and $\underline{u}_L := \underline{\mathbf{U}}_{L,0}$.

Plugging $u^\varepsilon_{L,\mathrm{app}}$ and $u^\varepsilon_{R,\mathrm{app}}$ in the viscous problem and identifying the terms with the same scale in ε, give the profiles equations: the function $\underline{u} := u_{R,-}\mathbf{1}_{\Omega_R^-} + u_{R,+}\mathbf{1}_{\Omega_R^+} + u_{L,-}\mathbf{1}_{\Omega_L^-} + u_{L,+}\mathbf{1}_{\Omega_L^+}$ which is also the 0th order underlined profile is defined by the following well-posed problems:

$$\begin{cases} \partial_t u_{R,-} + \partial_x(a_R u_{R,-}) = f_{R,-}, & (t,x)\in\Omega_R^-,\\ \partial_t u_{L,+} + \partial_x(a_L u_{L,+}) = f_{L,+}, & (t,x)\in\Omega_L^+,\\ u_{L,+}|_{x=0} = u_{R,-}|_{x=0} = 0, \end{cases}$$

and $u_{R,+}$ [resp. $u_{L,-}$] satisfies the awaited Cauchy problem on the open set Ω_R^+ [resp. Ω_L^-]. If $j\in\mathbb{N}$ is an odd number we have both $U_{R,j}=0$ and $U_{L,j}=0$. On the other hand for $j\in\mathbb{N}^*$ an even number, the profiles are computed by induction on even numbers as follows:

$$\begin{cases} \partial_t\underline{\mathbf{U}}_{2n,R,-} + \partial_x(a_R\underline{\mathbf{U}}_{2n,R,-}) = \partial_x^2\underline{\mathbf{U}}_{2n-2,R,-}, & (t,x)\in\Omega_R^-,\\ \partial_t\underline{\mathbf{U}}_{2n,L,+} + \partial_x(a_L\underline{\mathbf{U}}_{2n,L,+}) = \partial_x^2\underline{\mathbf{U}}_{2n-2,L,+}, & (t,x)\in\Omega_L^+,\\ \begin{pmatrix}\underline{\mathbf{U}}_{2n,L,+}|_{x=0}\\ \underline{\mathbf{U}}_{2n,R,-}|_{x=0}\end{pmatrix} = M^{-1}\begin{pmatrix} 0\\ -(\partial_x\underline{\mathbf{U}}_{2n-2,R,-}|_{x=0} - \partial_x\underline{\mathbf{U}}_{2n,L,+}|_{x=0})\end{pmatrix} \end{cases}$$

where

$$M := \begin{pmatrix} 1 & -1\\ a_L|_{x=0} & -a_R|_{x=0}\end{pmatrix};$$

this matrix is nonsingular since $a_L|_{x=0} - a_R|_{x=0} < 0$, and $\underline{\mathbf{U}}_{2n,R,+}$ [resp. $\underline{\mathbf{U}}_{2n,L,-}$] is the solution of Cauchy problem on the open set Ω_R^+ [resp. Ω_L^-] with a vanishing as initial data and $\partial_x^2\underline{\mathbf{U}}_{2n-2,R,+}$ [resp. $\partial_x^2\underline{\mathbf{U}}_{2n-2,L,-}$] as source term. In conclusion, all the profiles $\underline{\mathbf{U}}_n$ have now been constructed inductively.

We turn to the construction of the characteristic boundary layer profiles. Since their construction is analogous on each side of $\{x=0\}$, we will restrict ourselves to the construction of the profiles $U^c_{R,j,\pm}(t,\theta_R)$. The notation $[\theta]_\Gamma$ will be used for the jump of θ through the hypersurface Γ.

Before going further, we recall that $x_R(t)$ is the unique x such that $(t,x)\in\Gamma_R$. Because $u^\varepsilon_{\mathrm{app}}$ belongs to $C^1((0,T)\times\mathbb{R}^*)$, for all $0\leqslant j\leqslant M$, we have:

$$[U^c_{R,j}]_R = -[\underline{\mathbf{U}}_{R,j}]_{\Gamma_R}.$$

Let $[\underline{\mathbf{U}}_{R,j}]_{\Gamma_R}$ be given, for all $t\in(0,T)$, by:

$$[\underline{\mathbf{U}}_{R,j}]_{\Gamma_R}(t) = \lim_{x\to x_R(t),\,x>x_R(t)}\underline{\mathbf{U}}_{R,j,+}(t,x) - \lim_{x\to x_R(t),\,x<x_R(t)}\underline{\mathbf{U}}_{R,j,-}(t,x)$$

and $[U^c_{R,j}]_R$ be defined, for all $t\in(0,T)$, by:

$$[U^c_{R,j}]_R(t) = \lim_{\theta_R\to 0^+}U^c_{R,j,+}(t,\theta_R) - \lim_{\theta_R\to 0^-}U^c_{R,j,-}(t,\theta_R).$$

For all $0\leqslant j\leqslant M$, taking as a convention that the profiles indexed with a negative subscript vanishes, the profiles $U^c_{R,j,+}$ and $U^c_{R,j,-}$ are given by:

$$\begin{cases} \partial_t U^c_{R,j,+} - (\partial_x\varphi_R)^2\partial^2_{\theta_R}U^c_{R,j,+} + (\partial_x a_R)U^c_{R,j,+} = (\partial_x^2\varphi_R)\partial_{\theta_R}U^c_{R,j-1,+}, & t\in(0,T),\ \{\theta_R>0\},\\ \partial_t U^c_{R,j,-} - (\partial_x\varphi_R)^2\partial^2_{\theta_R}U^c_{R,j,-} + (\partial_x a_R)U^c_{R,j,-} = (\partial_x^2\varphi_R)\partial_{\theta_R}U^c_{R,j-1,-}, & t\in(0,T),\ \{\theta_R<0\},\\ [U^c_{R,j}]_R(t) = -[\underline{\mathbf{U}}_{R,j}]_{\Gamma_R}, \quad \forall t\in(0,T),\\ [\partial_{\theta_R}U^c_{R,j}]_R(t) = -\dfrac{1}{2(\partial_x\varphi)|_{x=x_R(t)}}\left([\partial_x\underline{\mathbf{U}}_{R,j-1}(t)]_{\Gamma_R}(t) + [\partial_x U^c_{R,j-1}(t)]_R\right), \quad \forall t\in(0,T),\\ U^c_{R,j,+}|_{t=0} = 0,\\ U^c_{R,j,-}|_{t=0} = 0. \end{cases}$$

The equations just described are well-posed since, by a change of unknowns, they can be formulated as classical heat equations.

We will now sketch the proof of the stability estimates.

We define the error $w^\varepsilon := u^\varepsilon_{\mathrm{app}} - u^\varepsilon$. By linearity w^ε is the solution of $P(L^\varepsilon, \varepsilon^M R^\varepsilon, 0)$; where R^ε belongs to $L^\infty([0,T] : L^2(\mathbb{R}))$. Multiplying by the solution and integrating by parts separately for $\{x > 0\}$ and $\{x < 0\}$, then using the transmission conditions on the boundary and Gronwall lemma, we get:

$$\|w^\varepsilon\|^2_{L^2(\mathbb{R})}(t) \leqslant \frac{1}{2}\varepsilon^M \int_0^T \mathrm{e}^{C(t-s)} \|R^\varepsilon\|^2_{L^2(\mathbb{R})}(s)\,\mathrm{d}s.$$

Constructing the profiles up to order $M = 1$, we obtain that there is $c > 0$, independent of ε, such that:

$$\|w^\varepsilon\|^2_{L^\infty([0,T]:L^2(\mathbb{R}))} \leqslant c\varepsilon,$$

thus achieving our proof.

3. Some remarks and perspectives

The kind of method used here can also be employed for linear hyperbolic systems in conservative or nonconservative forms. For scalar equations, depending on the setting of the discontinuity (see Theorem 1.1), energy estimates are not obtained by the same means. For systems, stability estimates for the problem are much less straightforward to obtain. In general, we can proceed by construction of a pseudodifferential Kreiss-type symmetrizer [10,8] when a geometric condition of stability is checked for the viscously perturbed problem (uniform Evans condition). In the nonconservative framework, some partial results have been given for systems in [4] and [5]. On one hand, the case where there are only compressive and traversing modes in the discontinuities of the coefficients is rather well understood (results given in multi-D and for piecewise smooth coefficients in [5]). On the other hand, many questions remain unsolved when expansive modes are also present. One question concerns the construction of an approximate solution for 2×2 systems when an expansive mode is coupled with a compressive one. Another is to generalize the results of [4] to several space dimensions. This last question is very interesting as new phenomena appears in multi-D, when expansive modes are present as shown in [6], for a scalar case.

In both [5] and [4], in a precise framework, the author shows the existence, uniqueness and stability of a small viscosity solution for linear nonconservative hyperbolic problems with one fixed discontinuity of the coefficient. Remark that, in the case of systems, the selected small viscosity solution depends of the chosen viscous perturbation of the operator.

Proving Theorem 1.2, we expose the core of our approach without requirement of elaborate tools. We can hope of getting a better understanding of hyperbolic systems with discontinuous coefficients by observing which solution is selected through a viscous approach.

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