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Partial Differential Equations

Scaling-sharp dispersive estimates for the Korteweg–de Vries group

Raphaël Côte [a], Luis Vega [b]

[a] Centre de mathématiques Laurent-Schwartz, École polytechnique, 91128 Palaiseau cedex, France
[b] Departamento de Matemáticas, Universidad del País Vasco, Aptdo. 644, 48080 Bilbao, Spain

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Abstract

We prove weighted estimates on the linear KdV group, which are scaling sharp. This kind of estimates is in the spirit of that used to prove small data scattering for the generalized KdV equations. ***To cite this article: R. Côte, L. Vega, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Estimées dispersives invariantes d'échelle pour le groupe de Korteweg–de Vries. Nous prouvons des inégalités à poids pour le groupe linéaire de KdV, qui sont optimales vis-à-vis du changement d'échelle. Ce type d'inégalité suit l'esprit de celles utilisées pour montrer que les solutions des équations de KdV généralisées dont les données sont petites dispersent linéairement. ***Pour citer cet article : R. Côte, L. Vega, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

The purpose of this short Note is to give a simple proof of two dispersive estimates which are heavily used in the proof of small data scattering for the generalized Korteweg–de Vries equations [2].

The proof of these estimates can be easily extended to other dispersive equations.

Denote $U(t)$ the linear Korteweg–de Vries group, i.e. $v = U(t)\phi$ is the solution to

$$\begin{cases} v_t + v_{xxx} = 0, \\ v(t=0) = \phi, \end{cases} \quad \text{i.e.} \quad \widehat{U(t)\phi} = \mathrm{e}^{\mathrm{i}t\xi^3/3}\hat{\phi} \quad \text{or} \quad \big(U(t)\phi\big)(x) = \frac{1}{t^{1/3}}\int \mathrm{Ai}\left(\frac{x-y}{t^{1/3}}\right)\phi(y)\,\mathrm{d}y,$$

where Ai is the Airy function

$$\mathrm{Ai}(z) = \frac{1}{\pi}\int_0^\infty \cos\left(\frac{\xi^3}{3} + \xi z\right)\mathrm{d}\xi.$$

Theorem 1. *Let $\phi, \psi \in L^2$, such that $x\phi, x\psi \in L^2$. Then*

E-mail addresses: cote@math.polytechnique.fr (R. Côte), luis.vega@ehu.es (L. Vega).

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$$\left\|U(t)\phi\right\|_{L^\infty}^2 \leqslant 2\|\mathrm{Ai}\|_{L^\infty}^2 t^{-2/3}\|\phi\|_{L^2}\|x\phi\|_{L^2}, \tag{1}$$

$$\left\|U(t)\phi U(-t)\psi_x\right\|_{L^\infty} \leqslant Ct^{-1}\big(\|\phi\|_{L^2}\|x\psi\|_{L^2}+\|\psi\|_{L^2}\|x\phi\|_{L^2}\big). \tag{2}$$

Furthermore, the constant $2\|\mathrm{Ai}\|_{L^\infty}^2$ *in the first estimate is optimal.*

Remark 1. These estimates are often used with ϕ replaced by $U(-t)\phi$: denoting $J(t)=U(t)xU(-t)$, they take the form

$$\|\phi\|_{L^\infty}^2 \leqslant Ct^{-2/3}\|\phi\|_{L^2}\|J(t)\phi\|_{L^2}, \tag{3}$$

$$\|\phi\psi_x\|_{L^\infty} \leqslant Ct^{-1}\big(\|\phi\|_{L^2}\|J(-t)\psi\|_{L^2}+\|\psi\|_{L^2}\|J(t)\phi\|_{L^2}\big). \tag{4}$$

Proof. Due to a scaling argument (and representation in term of the Airy function), we are reduced to show that

$$\left\|U(1)\phi\right\|_{L^\infty} \leqslant C\|\phi\|_{L^2}\|x\phi\|_{L^2},$$

and similarly for the second inequality. Hence we consider

$$\big(U(1)\phi\big)(x)=\int \mathrm{Ai}(x-y)\phi(y)\,\mathrm{d}y,$$

and we recall that the Airy function satisfies $|\mathrm{Ai}(x)|\leqslant C(1+|x|)^{-1/4}$ and $|\mathrm{Ai}'(x)|\leqslant C(1+|x|)^{1/4}$. Then

$$\begin{aligned}\left|U(1)\phi\right|^2(x) &= \iint \mathrm{Ai}(x-y)\phi(y)\mathrm{Ai}(x-z)\bar\phi(z)\frac{y-z}{y-z}\,\mathrm{d}y\,\mathrm{d}z\\ &= \int \mathrm{Ai}(x-y)y\phi(y)\underbrace{\int \frac{\mathrm{Ai}(x-z)}{y-z}\bar\phi(z)\,\mathrm{d}z}_{\mathcal{H}_{z\mapsto y}(\mathrm{Ai}(x-z)\phi(z))(y)}\mathrm{d}y - \int \mathrm{Ai}(x-z)z\bar\phi(z)\underbrace{\int\frac{\mathrm{Ai}(x-y)}{y-z}\phi(y)\,\mathrm{d}y}_{-\mathcal{H}_{y\mapsto z}(\mathrm{Ai}(x-y)\phi(y))(z)}\mathrm{d}z\\ &= 2\Re\int \mathrm{Ai}(x-y)y\phi(y)\mathcal{H}_{z\mapsto y}\big(\mathrm{Ai}(x-z)\phi(z)\big)(y)\,\mathrm{d}y,\end{aligned}$$

where $\mathcal{H}$ denotes the Hilbert transform (and with the slight abuse of notation $\frac{1}{x}$ for $\mathrm{vp}(\frac{1}{x})$). As $\mathcal{H}: L^2\to L^2$ is isometric and hence continuous (with norm 1), and $\mathrm{Ai}\in L^\infty$, we get

$$\left|U(1)\phi\right|^2(x) \leqslant 2\left\|\mathrm{Ai}(x-y)y\phi(y)\right\|_{L^2(\mathrm{d}y)}\left\|\mathcal{H}\big(\mathrm{Ai}(x-\cdot)\phi\big)(y)\right\|_{L^2(\mathrm{d}y)} \tag{5}$$

$$\leqslant 2\|\mathrm{Ai}\|_{L^\infty}^2\|y\phi\|_{L^2}\|\phi\|_{L^2}. \tag{6}$$

This is the first inequality. Let us now prove that the constant is sharp.

First consider the minimizers in the following Cauchy–Schwarz inequality:

$$\left|\int y\psi(y)\mathcal{H}(\psi)(y)\,\mathrm{d}y\right| \leqslant \left\|y\psi(y)\right\|_{L^2(dy)}\|\psi\|_{L^2}. \tag{7}$$

There is equality if $y\psi(y)=\lambda\mathcal{H}(\psi)(y)$ for some $\lambda\in\mathbb{C}$. Then a Fourier Transform shows that $\partial_\xi\hat\psi(\xi)=\lambda\,\mathrm{sgn}\,\xi\,\hat\psi(\xi)$, hence $\hat\psi(\xi)=C\exp(-\lambda|x|)$, or equivalently, one has equality in (7) as soon as

$$\psi(y)=\frac{C}{1+(y/\lambda)^2}\quad \text{for some } \lambda, C\in\mathbb{C}. \tag{8}$$

(Notice that all the functions involved lie in L^2.)

We now go back to (6). Let $x_0\in\mathbb{R}$ where $|\mathrm{Ai}|$ reaches its maximum. Now as $\mathrm{Ai}(x_0)\neq 0$, let $\varepsilon>0$ such that for all $y\in[-\varepsilon,\varepsilon]$, $|\mathrm{Ai}(x_0-y)|\geqslant|\mathrm{Ai}(x_0)|/2$, and consider the sequence of functions

$$\phi_n(x)=\frac{\sqrt{n}}{1+(ny)^2}\frac{\mathbb{1}_{|y|\leqslant\varepsilon}}{\mathrm{Ai}(x_0-y)}.$$

Denote $\psi_n(y)=\frac{\mathbb{1}_{|y|\leqslant n\varepsilon}}{1+y^2}$. As $\mathrm{Ai}(x_0-y)\phi_n(ny)=\sqrt{n}\psi_n(ny)$,

$$\left|U(1)\phi_n\right|^2(x_0)=2\int y\sqrt{n}\psi_n(ny)\mathcal{H}_{z\to y}\big(\sqrt{n}\psi_n(nz)\big)(y)\,\mathrm{d}y=\frac{2}{n}\int y\psi_n(ny)\mathcal{H}(\psi_n)(ny)\,\mathrm{d}y.$$

One easily sees that $\psi_n(y) \to \frac{1}{1+|y|^2}$ in L^2 and $y\psi_n(y) \to \frac{y}{1+|y|^2}$ in L^2, and hence, in view of (8), as $\mathcal{H}$ is homogeneous of degree 0 and L^2 isometric, we have

$$\begin{aligned}\big|U(1)\phi_n\big|^2(x_0) &\sim \frac{2}{n}\int \frac{y}{1+|y|^2}\mathcal{H}\bigg(\frac{1}{1+|\cdot|^2}\bigg)(y)\,\mathrm{d}y \sim \frac{2}{n}\bigg\|\frac{y}{1+|y|^2}\bigg\|_{L^2}\bigg\|\frac{1}{1+|y|^2}\bigg\|_{L^2}\\ &\sim 2\big\|y\sqrt{n}\psi_n(ny)\big\|_{L^2}\|\sqrt{n}\psi_n(ny)\|_{L^2}\\ &\sim 2\big\|y\mathrm{Ai}(x_0-y)\phi_n(y)\big\|_{L^2}\big\|\mathrm{Ai}(x_0-y)\phi_n(y)\big\|_{L^2}.\end{aligned}$$

As ϕ_n concentrates at point 0, we deduce

$$\big|U(1)\phi_n\big|^2(x_0) \sim 2\big|\mathrm{Ai}(x_0)\big|^2\big\|y\phi_n(y)\big\|_{L^2}\big\|\phi_n(y)\big\|_{L^2} \quad \text{as } n\to\infty, \tag{9}$$

which proves that the sharp constant in the first inequality is $2\|\mathrm{Ai}\|_{L^\infty}^2$.

For the second inequality (estimate of the derivative), we have as for the first inequality:

$$\begin{aligned}&\big(U(1)\phi U(1)\bar{\psi}_x\big)(x)\\ &\quad= \iint \mathrm{Ai}(x-y)\phi(y)\mathrm{Ai}'(x-z)\bar{\psi}(z)\frac{y-z}{y-z}\,\mathrm{d}y\,\mathrm{d}z\\ &\quad= \int \mathrm{Ai}'(x-z)\bar{\psi}(z)\bigg(\int\frac{\mathrm{Ai}(x-y)y\phi(y)}{y-z}\,\mathrm{d}y\bigg)\mathrm{d}z - \int \mathrm{Ai}'(x-z)z\bar{\psi}(z)\bigg(\int\frac{\mathrm{Ai}(x-y)\phi(y)}{z-y}\,\mathrm{d}y\bigg)\mathrm{d}z\\ &\quad= \int \mathrm{Ai}'(x-z)\bar{\psi}(z)\mathcal{H}_{y\mapsto z}\big(\mathrm{Ai}(x-y)y\phi(y)\big)(z)\,\mathrm{d}z - \int \mathrm{Ai}'(x-z)z\bar{\psi}(z)\mathcal{H}_{y\to z}\big(\mathrm{Ai}(x-y)\phi(y)\big)(z)\,\mathrm{d}z.\end{aligned}$$

Denote $\omega_x(z) = \frac{1}{\sqrt{1+|x-z|}}$: $\omega_x^{-1} \in A_2$ (with the notation of [4]), so that there exists C not depending on x such that

$$\forall v, \quad \int |\mathcal{H}v|^2\omega_x^{-1} \leqslant C\int |v|^2\omega_x^{-1}.$$

Recall the well-known asymptotic $|\mathrm{Ai}'(x)| \leqslant C(1+|x|^{1/4})$. Then

$$\begin{aligned}&\bigg|\int \mathrm{Ai}'(x-z)\bar{\psi}(z)\mathcal{H}_{y\to z}\big(\mathrm{Ai}(x-y)y\phi(y)\big)(z)\,\mathrm{d}z\bigg|\\ &\quad\leqslant \bigg(\int\big|\mathrm{Ai}'(x-z)\bar{\psi}(z)\big|^2\omega_x(z)\,\mathrm{d}y\bigg)^{1/2}\bigg(\int\big|\mathcal{H}_{y\to z}\big(\mathrm{Ai}(x-y)y\phi(y)\big)(z)\big|^2\omega_x^{-1}(z)\,\mathrm{d}z\bigg)^{1/2}\\ &\quad\leqslant C\|\psi\|_{L^2}\bigg(\int\big|\mathrm{Ai}(x-y)y\phi(y)\big|^2\omega_x^{-1}(y)\,\mathrm{d}y\bigg)^{1/2}\\ &\quad\leqslant C\|\psi\|_{L^2}\big\|y\phi(y)\big\|_{L^2}.\end{aligned}$$

In the same way,

$$\begin{aligned}&\bigg|\int \mathrm{Ai}'(x-z)z\bar{\psi}(z)\mathcal{H}\big(\mathrm{Ai}(x-\cdot)\phi\big)(z)\,\mathrm{d}z\bigg|\\ &\quad\leqslant \bigg(\int\big|\mathrm{Ai}'(x-z)z\bar{\psi}(z)\big|^2\omega_x(z)\,\mathrm{d}z\bigg)^{1/2}\bigg(\int\big|\mathcal{H}\big(\mathrm{Ai}(x-\cdot)\phi\big)(z)\big|^2\omega_x^{-1}(z)\,\mathrm{d}z\bigg)^{1/2}\\ &\quad\leqslant C\big\|z\psi(z)\big\|_{L^2}\bigg(\int\big|\mathrm{Ai}(x-y)\phi(y)\big|^2\omega_x^{-1}(y)\,\mathrm{d}y\bigg)^{1/2} \leqslant C\big\|y\psi(y)\big\|_{L^2}\|\phi\|_{L^2}.\end{aligned}$$

Thus:

$$\big\|U(1)\phi U(1)\bar{\psi}_x\big\|_{L^\infty} \leqslant C\big(\|\phi\|_{L^2}\|x\psi\|_{L^2} + \|\psi\|_{L^2}\|x\phi\|_{L^2}\big).$$

Up to scaling and replacing ψ by $\bar{\psi}$, this is the second inequality. □

Remark 2. This proof (especially (5)) is reminiscent of that in [3] (see also [1])

$$\|\phi\|_{L^\infty}^2 \leqslant \|\phi\|_{L^2}\|\phi'\|_{L^2},$$

where the constant is sharp and minimizers are $C\mathrm{e}^{-\lambda|x|}$. This has application to the Schrödinger group $\mathcal{U}(t)$ (i.e. $\widehat{\mathcal{U}(t)\phi} = \mathrm{e}^{\mathrm{i}t\xi^2}\hat{\phi}$). We have the following Schrödinger version of estimate (3) (notice that $\mathcal{U}(t)x\mathcal{U}(-t) = \mathrm{e}^{\frac{\mathrm{i}|x|^2}{4t}}\frac{\mathrm{i}t}{2}\partial_x\mathrm{e}^{-\frac{\mathrm{i}|x|^2}{4t}}$):

$$\|\psi\|_{L^\infty}^2 = \left\|\mathrm{e}^{\frac{\mathrm{i}|x|^2}{4t}}\psi\right\|_{L^\infty}^2 \leqslant \left\|\mathrm{e}^{\mathrm{i}\frac{|x|^2}{4t}}\psi\right\|_{L^2}\left\|\mathrm{e}^{\mathrm{i}\frac{|x|^2}{4t}}\partial_x\mathrm{e}^{\mathrm{i}\frac{|x|^2}{4t}}\psi\right\|_{L^2} \leqslant \frac{2}{t}\|\psi\|_{L^2}\left\|\mathcal{U}(t)x\mathcal{U}(-t)\psi\right\|_{L^2}.$$

From Theorem 1, we can easily obtain the optimal decay in a scaling sharp Besov like space. Let $\varphi \in \mathcal{D}(\mathbb{R})$ be non-negative with support in $]-2,2[$ and such that φ equals 1 in a neighbourhood of $[-1.5,1.5]$. Denote $\psi(x) = \varphi(2x) - \varphi(x)$ and $\psi_j(x) = \psi(x/2^j)$. Finally introduce

$$\|\phi\|_{N_t} = \sum_{j=-\infty}^{\infty} 2^{j/2}\left\|\psi_j U(-t)\phi\right\|_{L^2}.$$

Corollary 2. *We have*:

$$\|\phi\|_{L^\infty} \leqslant Ct^{-1/3}\|\phi\|_{N_t}.$$

Proof. Notice that $|x\psi_j(x)| \leqslant 2^{j+1}\psi_j(x)$. As $\phi = \sum_j U(t)\psi_j U(-t)\phi$, we have:

$$\begin{aligned}\|\phi\|_{L^\infty} &\leqslant \sum_j \left\|U(t)\psi_j U(-t)\phi\right\|_{L^\infty} \leqslant Ct^{-1/3}\sum_j \left\|U(t)\psi_j U(-t)\phi\right\|_{L^2}^{1/2}\left\|U(t)xU(-t)U(t)\psi_j U(-t)\phi\right\|_{L^2}^{1/2}\\ &\leqslant Ct^{-1/3}\sum_j \left\|U(t)\psi_j U(-t)\phi\right\|_{L^2}^{1/2}\left\|x\psi_j U(-t)\phi\right\|_{L^2}^{1/2}\\ &\leqslant Ct^{-1/3}\sum_j \left\|U(t)\psi_j U(-t)\phi\right\|_{L^2}^{1/2} 2^{j/2}\left\|U(t)\psi_j U(-t)\phi\right\|_{L^2}^{1/2} \leqslant Ct^{-1/3}\|\phi\|_{N_t}.\end{aligned}$$

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