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Mathematical Analysis

Baire category and zero sets

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Abstract

Quasi-all continuous functions have a zero set which is a perfect Kronecker set with Hausdorff dimension zero. ***To cite this article: T. Körner, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Ensembles de zéros et catégorie de Baire. Quasi-surement, les zéros d'une fonction continue forment un ensemble de Kronecker parfait de dimension d'Haudorff zéro. ***Pour citer cet article : T. Körner, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

1. Introduction

We work on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and write $C(\mathbb{T})$ for the space of continuous functions $f : \mathbb{T} \to \mathbb{R}$. The zero set Z_f of such a function is defined by

$$Z_f = \{t \in \mathbb{T}:\ f(t) = 0\}.$$

The zero sets of random functions have been extensively studied (see for example [1] Chapter 14). Kahane asks a natural question. What can be said if we replace probabilistic by Baire category considerations?

Theorem 1.1. *Let $h : [0,1] \to [0,\infty)$ be continuous and strictly increasing with $h(0) = 0$. Consider the space of real continuous functions $C(\mathbb{T})$ under the uniform norm. Quasi-all $f \in C(\mathbb{T})$ have a zero set which is a perfect Kronecker set of Hausdorff h-measure* 0.

(Note that the empty set is perfect, Kronecker and of Hausdorff h-measure 0.)

In fact we can prove a stronger result:

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Lemma 1.2. *Let A_q be set of first category in $\mathbb{T}^q$ with the usual Euclidean metric $[q \geqslant 1]$ and let*

$$A = \bigcup_{q=1}^{\infty} \{\mathbf{x} \in \mathbf{T}^{\mathbb{N}}\colon\ (x_1, x_2, \dots, x_q) \in A_q\}.$$

Consider the space of real continuous functions $C(\mathbb{T})$ under the uniform norm. Quasi-all $f \in C(\mathbb{T})$ have a zero set which is a perfect Kronecker set and satisfies the following condition: If $x_1, x_2, \dots$ are distinct zeros of f, then $\mathbf{x} \notin A$.

To see how Lemma 1.2 can be used, we prove the following result:

Lemma 1.3. *Let $h : [0, 1] \to [0, \infty)$ be continuous and strictly increasing with $h(0) = 0$. Quasi-always, the group generated by Z_f has Hausdorff h-measure zero.*

To obtain Lemma 1.3 from Lemma 1.2 we argue as follows. We can find closed nowhere dense sets B_u such that $\mathbb{T} \setminus \bigcup_{u=1}^{\infty} B_u$ has Hausdorff h-measure zero. If

$$\mathbf{m} = (m_1, m_2, \dots, m_q) \in \Lambda(q) = (\mathbb{Z} \setminus \{\mathbf{0}\})^q$$

and $r \geqslant q + 2$, we write $B(q, \mathbf{m}, r, u)$ for the set of $\mathbf{x} \in \mathbb{T}^q$ such that $|x_i - x_j| \geqslant 1/r$ for $i \neq j$ and $\sum_{j=1}^{q} m_j x_j \in B_u$. A Bolzano–Weierstrass argument shows that $B(q, \mathbf{m}, r, u)$ is closed and the fact that B_u is nowhere dense shows that $B(q, \mathbf{m}, r, u)$ is nowhere dense. Thus

$$A_q = \bigcup_{u=1}^{\infty} \bigcup_{\mathbf{m} \in \Lambda(q)} \bigcup_{r=q+2}^{\infty} B(q, \mathbf{m}, r, u)$$

is of first category. The group generated by the complement of A lies in $(\mathbb{T} \setminus \bigcup_{q=1}^{\infty} B_q) \cup \{0\}$ and so has Hausdorff h-measure zero.

The same ideas are used in deeper context in [2].

2. Proof of Theorem 1.1

We start by showing that, quasi-always, Z_f is perfect, that is to say, contains no isolated points.

Lemma 2.1.

(i) *If $m \geqslant r \geqslant 1$, let $\mathcal{E}_{m,r}$ be the set of $f \in C(\mathbb{T})$ such that the interval $[(r-1)/m, r/m]$ has at least one of the following three properties.*
  (a) *$Z_f \cap [(r-1)/m, r/m]$ is empty.*
  (b) *We can find $(r-1)/m \leqslant x_1 < x_2 < x_3 \leqslant r/m$ with $f(x_1), f(x_3) < 0 < f(x_2)$.*
  (c) *We can find $(r-1)/m \leqslant x_1 < x_2 < x_3 \leqslant r/m$ with $f(x_1), f(x_3) > 0 > f(x_2)$.*
  $\mathcal{E}_{m,r}$ is open and dense.

(ii) *Quasi-always, Z_f contains no isolated points.*

The proof of part (i) is routine. To obtain (ii) we observe that, if Z_f contains an isolated point, we can find an m and an $1 \leqslant r \leqslant m$ such that $f \notin \mathcal{E}_{m,r}$.

Lemma 2.2. *Using the notation and hypotheses of Lemma 1.2, quasi-all $f \in C(\mathbb{T})$ have the property that, if $x_1, x_2, \dots$ are distinct zeros of f, then $\mathbf{x} \notin A$.*

By a standard category argument, it suffices to show that, quasi-always, Z_f has the property that, whenever x_1, $x_2, \dots, x_q$ are distinct zeros of f, then $\mathbf{x} = (x_1, x_2, \dots, x_q) \notin A_q$. The same argument shows that we may suppose $A_q = F_q$ a closed nowhere dense subset of $\mathbb{T}^q$.

If $n \geqslant q + 2$, let write $\mathcal{G}_n$ for the set of $f \in C(\mathbb{T})$ with the property that, whenever $\mathbf{x} \in \mathbb{T}^q$ satisfies $|x_i - x_j| \geqslant 1/n$, we have $\mathbf{x} \in F_q$. A Bolzano–Weierstrass argument establishes that the complement of $\mathcal{G}_n$ is closed and so $\mathcal{G}_n$ is open.

On the other hand, if $\epsilon > 0$ and $f \in C(\mathbb{T})$, we can find a piecewise linear, nowhere locally constant $g \in C(\mathbb{T})$ with $\|g - f\|_\infty < \epsilon/2$. Since Z_g is finite, we can use the fact that F_q is nowhere dense to find a piecewise linear, nowhere locally constant $h \in C(\mathbb{T})$ with $\|g - h\|_\infty < \epsilon/2$ such that $h \in \mathcal{G}_n$. Thus $\mathcal{G}_n$ is open and everywhere dense. The required result follows by considering $\bigcap_{n=q+2}^\infty \mathcal{G}_n$.

Finally we prove:

Lemma 2.3. *Using the notation and hypotheses of Lemma* 1.2*, quasi-all $f \in C(\mathbb{T})$ have a zero set which is Kronecker.*

Theorem 9 of [2] shows that we cannot deduce Lemma 2.3 from Lemma 2.2.

We recall two definitions:

Definition 2.4.

(i) We write $S(\mathbb{T})$ for the set of $\phi \in C(\mathbb{T})$ such that $|\phi(t)| = 1$ for all $t \in \mathbb{T}$.
(ii) A closed set E in $\mathbb{T}$ is said to be *Kronecker* if, given $\phi \in S(\mathbb{T})$ and $\epsilon > 0$, we can find an $N \in \mathbb{Z}$ such that $|\exp(2\pi \mathrm{i} Nt) - \phi(t)| < \epsilon$ for all $t \in E$.

Standard arguments show that $S(\mathbb{T})$ has a countable dense subset $\phi_1, \phi_2, \ldots$ where, for convenience, we suppose each term ϕ_m occurs infinitely often. If we let $\mathcal{H}_m$ be the set of $f \in C(\mathbb{T})$ such that there exists a character $N \in \mathbb{Z}$ with

$$\left|\exp(2\pi \mathrm{i} Nt) - \phi_m(t)\right| < 1/m$$

for all $t \in Z_f$, then it is easy to see that $\mathcal{H}_m$ is open.

To show that $\mathcal{H}_m$ is dense, observe that, given $\epsilon > 0$ and $f \in C(\mathbb{T})$, we can find a piecewise linear, nowhere locally constant $g \in C(\mathbb{T})$ with $\|h - g\|_\infty < \epsilon/2$ and then a piecewise linear, nowhere locally constant $h \in C(\mathbb{T})$ with Z_h finite and independent and so, by Kronecker's theorem, such that there exists a integer N with

$$\left\|\exp(2\pi \mathrm{i} Nt) - \phi_n(t)\right\| < 1/m$$

for all $t \in Z_g$. Since $\|h - f\|_\infty < \epsilon$ and $h \in \mathcal{H}_m$ we are done.

It follows that, quasi-always, $f \in \bigcap_{m=1}^\infty \mathcal{H}_m$ and so Z_f is Kronecker.

Since probabilistic methods often produce functions of a certain Lipschitz class, it may be worth remarking that the same proofs give the following result:

Lemma 2.5. *Suppose $k : [0, 1] \to [0, \infty)$ is a continuous strictly increasing function with $k(0) = 0$ and $k(t)/t \to \infty$ as $t \to 0+$. Let Λ be the space of $f \in C(\mathbb{T})$ such that*

$$\|f\|_\Lambda = \|f\|_\infty + \sup_{s \neq t} \frac{|f(s) - f(t)|}{k(|s - t|)}$$

is finite. The space Λ of continuous functions with the norm $\|.\|_\Lambda$ is complete. Let A_q and A be as in Lemma 1.2*. Quasi-all $f \in \Lambda$ have a zero set which is a perfect Kronecker set and satisfies the following condition. If $x_1, x_2, \ldots$ are distinct zeros of f, then $\mathbf{x} \notin A$.*

References

[1] J.P. Kahane, Some Random Series of Functions, second edition, Cambridge Studies in Advanced Mathematics, vol. 5, Cambridge University Press, Cambridge, 1985.
[2] T.W. Körner, Variations on a theme of Debs and Saint Raymond, J. LMS, in press.