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Differential Geometry

On the group of symplectic homeomorphisms

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Abstract

Let (M, ω) be a closed symplectic manifold. We define a Hofer-like metric d on the identity component $\mathrm{Sym}(M, \omega)_0$ in the group $\mathrm{Symp}(M, \omega)$ of all symplectic diffeomorphisms of (M, ω). Unlike the Hofer metric on the group $\mathrm{Ham}(M, \omega)$ of Hamiltonian diffeomorphisms, the metric d is not bi-invariant. We show that the metric topology τ defined by d is natural (i.e. independent of the choice involved in its definition). We define the *symplectic topology* as a blend of the Hofer-like topology τ and the $\mathcal{C}^0$-topology. We use it to construct a subgroup $\mathrm{SSympeo}(M, \omega)$ of the group $\mathrm{Sympeo}(M, \omega)$ of all symplectic homeomorphisms, containing the group $\mathrm{Hameo}(M, \omega)$ of Hamiltonian homeomorphisms (introduced by Oh and Muller). If M is simply connected $\mathrm{SSympeo}(M, \omega)$ coincides with $\mathrm{Hameo}(M, \omega)$. Moreover its commutator subgroup $[\mathrm{SSympeo}(M, \omega), \mathrm{SSympeo}(M, \omega)]$ is contained in $\mathrm{Hameo}(M, \omega)$. ***To cite this article: A. Banyaga, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Sur le groupe des homéomorphismes symplectiques. Soit (M, ω) une variété symplectique fermée. On définit à la Hofer une métrique d sur la composante connexe de l'identité dans le groupe $\mathrm{Symp}(M, \omega)$ de tous les difféomorphismes symplectiques. Contrairement à la métrique de Hofer, la métrique d n'est pas bi-invariante. Nous montrons que la topologie métrique τ définie par d est naturelle (i.e. indépendante des choix faits pour la définir). Nous définissons la *topologie symplectique* comme une combinaison de la topologie τ et de la $\mathcal{C}^0$-topologie. Nous l'utilisons pour construire un sous-groupe $\mathrm{SSympeo}(M, \omega)$ du groupe $\mathrm{Sympeo}(M, \omega)$ des homéomorphismes symplectiques, qui contient le groupe $\mathrm{Hameo}(M, \omega)$ des homéomorphismes hamiltoniens (introduits par Oh et Muller). Si M est simplement connexe, $\mathrm{SSympeo}(M, \omega)$ coïncide avec $\mathrm{Hameo}(M, \omega)$. De plus, son sous-groupe des commutateurs $[\mathrm{SSympeo}(M, \omega), \mathrm{SSympeo}(M, \omega)]$ est contenu dans $\mathrm{Hameo}(M, \omega)$. ***Pour citer cet article : A. Banyaga, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

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Sur l'espace $\mathrm{Iso}(M, \omega)$ des isotopies symplectiques d'une variété symplectique fermée (M, ω), on introduit la « distance symplectique » d_{symp} comme suit : on fixe une métrique riemannienne g et on considère la distance induite d_0. Pour toute isotopie $\Phi = (\phi_t) \in \mathrm{Iso}(M, \omega)$, on considère la décomposition de Hodge de la 1-forme fermée $i(\dot{\phi}_t)\omega$: $i(\dot{\phi}_t)\omega = H_t^{\Phi} + \mathrm{d}u_t^{\Phi}$ où H_t^{Φ} est une forme harmonique. Si $\Phi, \Psi \in \mathrm{Iso}(M, \omega)$, on pose

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$$D_0(\Phi,\Psi)=\int_0^1\left(\left|H_t^{\Phi}-H_t^{\Psi}\right|+\operatorname{osc}\left(u_t^{\Phi}-u_t^{\Psi}\right)\right)\mathrm{d}t,$$

où $|\cdot|$ est la norme sur l'espace des formes harmoniques décrite dans le paragraphe 3 de la version anglaise et $\operatorname{osc}(F)=\max_{x\in M}F(x)-\min_{x\in M}F(x)$ est l'oscillation d'une fonction F sur M.

Si $\Phi^{-1}=(\phi_t^{-1})$ et $\Psi^{-1}=(\psi_t^{-1})$ dénotent les isotopies inverses, on pose :

$$D(\Phi,\Psi)=\left(D_0(\Phi,\Psi)+D_0\left(\Phi^{-1},\Psi^{-1}\right)\right)/2.$$

On définit la *distance symplectique* sur $\mathrm{Iso}(M,\omega)$ par :

$$d_{\mathrm{symp}}(\Phi,\Psi)=\bar{d}(\Phi,\Psi)+D(\Phi,\Psi),$$

où

$$\bar{d}(\Phi,\Psi)=\sup_{t\in[0,1]}\left[\max\left(\sup_{x\in M}d_0\left(\phi_t(x),\psi_t(x)\right),\sup_{x\in M}d_0\left(\phi_t^{-1}(x),\psi_t^{-1}(x)\right)\right)\right].$$

La « *topologie symplectique* » sur $\mathrm{Iso}(M,\omega)$ est la topologie métrique induite par la distance d_{symp}. Elle se réduit à la topologie hamiltonienne d'Oh–Muller sur l'espace des isotopies hamiltoniennes [7].

On annonce les résultats suivants :

Théorème 0.1. *La topologie symplectique sur* $\mathrm{Iso}(M,\omega)$ *est naturelle* (*indépendante des choix faits pour la définir*).

Théorème 0.2. *Soit* $\mathrm{SSympeo}(M,\omega)$ *l'ensemble de tous les homéomorphismes* h *de* M *tels qu'il existe un chemin continu* $\lambda:[0,1]\to\mathrm{Homeo}(M)$ *tel que* $\lambda(0)=\mathrm{Id}$ *et* $\lambda(1)=h$, *et une suite de Cauchy* $\Phi_n\in\mathrm{Iso}(M,\omega)$ (*pour la distance* d_{symp}) *qui converge* C^0 *vers* λ (*dans la topologie* $\bar{d}$). *Alors,* $\mathrm{SSympeo}(M,\omega)$ *est un groupe topologique, qui contient le groupe* $\mathrm{Hameo}(M,\omega)$ (*décrit au paragraphe* 2)*, et qui se réduit à ce dernier si* M *est simplement connexe. De plus,* $[\mathrm{SSympeo}(M,\omega),\mathrm{SSympeo}(M,\omega)]\subset\mathrm{Hameo}(M,\omega)$.

Conjecture 0.1. *Soit* $Sympeo_0(M,\omega)$ *la composante connexe de l'identité dans* $\mathrm{Sympeo}(M,\omega)$. *L'inclusion* $\mathrm{SSympeo}(M,\omega)\subset Sympeo_0(M,\omega)$ *est stricte.*

Cette conjecture implique la conjecture d'Oh–Muller [7] qui dit que la composante connexe de l'identité $\mathrm{Homeo}_{\Omega}(S^2)_0$ dans le groupe des homéomorphismes de la sphère qui préservent la mesure Ω (définie par ω) n'est pas un groupe simple.

La longueur $l(\Phi)$ d'une isotopie symplectique $\Phi=(\phi_t)$ est la distance $D(\Phi,\mathbf{1})$ où $\mathbf{1}$ est l'isotopie $\phi_t=\mathrm{Id}$.

Théorème 0.3. *Soit* (M,ω) *une variété symplectique fermée. Pour tout* $\phi\in\mathrm{Symp}(M,\omega)$*, on pose* :

$$e(\phi)=\inf\left(l(\Phi)\right)$$

où l' infimum est pris sur toutes les isotopies symplectiques Φ *de* ϕ *à l'identité. L'application* $d:\mathrm{Symp}(M,\omega)_0\times\mathrm{Symp}(M,\omega)_0\to\mathbb{R}\cup\{\infty\}$, $(\psi,\phi)\mapsto d(\phi,\psi)=e(\phi\psi^{-1})$ *est une métrique* (*non bi-invariante*) *sur la composante connexe de l'identité* $\mathrm{Symp}(M,\omega)_0$ *dans le groupe* $\mathrm{Symp}(M,\omega)$ *de tous les difféomorphismes symplectiques. La topologie métrique induite par* d *sur* $\mathrm{Symp}(M,\omega)_0$ *est naturelle* (*indépendante des choix faits pour la définir*).

Notons que, si $\phi\in\mathrm{Ham}(M,\omega)$, on a $e(\phi)\leqslant\|\phi\|_H$, où $\|\cdot\|_H$ est la norme de Hofer (voir (2)).

1. Introduction

Let (M,ω) be a closed symplectic manifold. The Eliashberg–Gromov symplectic rigidity theorem says that the group $\mathrm{Symp}(M,\omega)$ of symplectomorphisms is C^0 closed in the group $\mathrm{Diff}^{\infty}(M)$ of C^{∞} diffeomorphisms of M (see [5]). This means that the "symplectic" nature of a sequence of symplectomorphisms survives topological limits. This is an evidence that there is a C^0 *symplectic topology* underlying the symplectic geometry of a symplectic manifold (M,ω).

According to Oh–Muller [7], the automorphism group of the C^0 symplectic topology is the closure:

$$\overline{\mathrm{Symp}(M, \omega)} \subset \mathrm{Homeo}(M)$$

of the group $\mathrm{Symp}(M, \omega)$ of all symplectomorphisms of (M, ω) in the group $\mathrm{Homeo}(M)$ of homeomorphisms of M endowed with the C^0 topology. That group is denoted by $\mathrm{Sympeo}(M, \omega)$ and is called the group of symplectic homeomorphisms. Previously, Hofer (and others) had defined the automorphism group of the C^0 symplectic topology as the set of homeomorphisms of the symplectic manifolds (M, ω) which preserve a symplectic capacity. It is not clear how these two groups relate.

In this Note, we define a subgroup of $\mathrm{Sympeo}(M, \omega)$, denoted $\mathrm{SSympeo}(M, \omega)$ and nicknamed the group of strong symplectic homeomorphisms, using a blend of the C^0 topology and a Hofer-like topology on the space $\mathrm{Iso}(M, \omega)$ of symplectic isotopies of (M, ω), the same way Oh–Muller [7] defined the group $\mathrm{Hameo}(M, \omega)$ of Hamiltonian homeomorphisms of (M, ω).

2. The group of Hamiltonian homeomorphisms [7]

The C^0 topology on $\mathrm{Homeo}(M)$ coincides with the metric topology coming from the metric

$$\bar{d}(g, h) = \max\Big(\sup_{x \in M} d_0\big(g(x), h(x)\big), \sup_{x \in M} d_0\big(g^{-1}(x), h^{-1}(x)\big)\Big)$$

where d_0 is a distance on M induced by some riemannian metric [7].

On the space $\mathrm{PHomeo}(M)$ of continuous paths $\gamma : [0, 1] \to \mathrm{Homeo}(M)$, one has the distance

$$\bar{d}(\gamma, \mu) = \sup_{t \in [0,1]} \bar{d}\big(\gamma(t), \mu(t)\big).$$

Consider the space $\mathrm{PHam}(M)$ of all isotopies $\Phi_H = [t \mapsto \Phi_H^t]$ where Φ_H^t is the family of Hamiltonian diffeomorphisms obtained by integration of the family of vector fields X_H for a smooth family of real functions $H(x, t)$ on M.

Recall that X_H is uniquely defined by the equation $i(X_H)\omega = dH$ where $i(\cdot)$ is the interior product. The group $\mathrm{Ham}(M, \omega)$ is the set of all time-one maps $\{\Phi_H^1\}$.

Definition 2.1. (See [7].) *The Hamiltonian topology* on $P\,\mathrm{Ham}(M)$ is the metric topology defined by the distance

$$d_{\mathrm{ham}}(\Phi_H, \Phi_{H'}) = \|H - H'\| + \bar{d}(\Phi_H, \Phi_{H'})$$

where $\|H - H'\| = \int_0^1 \mathrm{osc}(H - H')\,\mathrm{d}t$, and the oscillation of a function u is

$$\mathrm{osc}(u) = \max_{x \in M} u(x) - \min_{x \in M} u(x).$$

Definition 2.2. (See [7].) A homeomorphism h is said to be a Hamiltonian homeomorphism of (M, ω) if there exists a continuous path $\lambda \in \mathrm{PHomeo}(M)$ such that $\lambda(0) = \mathrm{Id}$, $\lambda(1) = h$ and there exists a Cauchy sequence (for the d_{ham} distance) of Hamiltonian isotopies Φ_{H^n}, which C^0 converges to λ (in the $\bar{d}$ metric).

The main result of [7] is that the set $\mathrm{Hameo}(M, \omega)$ of all Hamiltonian homeomorphisms of a compact symplectic manifold (M, ω) is a normal subgroup of $\mathrm{Sympeo}(M, \omega)$.

This group is the topological analogue of the group $\mathrm{Ham}(M, \omega)$ of Hamiltonian diffeomorphisms. Almost nothing is known on the groups $\mathrm{Sympeo}(M, \omega)$ and $\mathrm{Hameo}(M, \omega)$.

3. The symplectic topology on Iso(M, ω) and the group SSympeo(M, ω)

Let $\mathrm{Iso}(M, \omega)$ denote the space of symplectic isotopies of a compact symplectic manifold (M, ω). Recall that a symplectic isotopy is a smooth map $\Phi : M \times [0, 1] \to M$ such that for all $t \in [0, 1]$, $\phi_t : M \to M$, $x \mapsto \Phi(x, t)$ is a symplectic diffeomorphism and $\phi_0 = \mathrm{Id}$.

The "Lie algebra" of $\mathrm{Symp}(M, \omega)$ is the space $\mathrm{sym}(M, \omega)$ of symplectic vector fields, i.e the set of vector fields X such that $i_X\omega$ is a closed form.

Let ϕ_t be a symplectic isotopy, then

$$\dot{\phi}_t(x) = \frac{\mathrm{d}\phi_t}{\mathrm{d}t}\left(\phi_t^{-1}(x)\right)$$

is a smooth family of symplectic vector fields.

By the theorem of existence and uniqueness of solutions of ODE's,

$$\Phi \in \mathrm{Iso}(M,\omega) \mapsto \dot{\phi}_t$$

is a 1-1 correspondence between $\mathrm{Iso}(M,\omega)$ and the space $C^\infty([0,1], \mathrm{symp}(M,\omega))$ of smooth families of symplectic vector fields. Hence any distance on $C^\infty([0,1], \mathrm{symp}(M,\omega))$ gives rise to a distance on $\mathrm{Iso}(M,\omega)$.

We define a norm $\|\cdot\|$ on $\mathrm{symp}(M,\omega)$ as follows: first we fix a riemannian metric g (which may be the one we used to define d_0 above, or any other riemannian metric), and a basis $\mathbf{B} = \{h_1,\ldots,h_k\}$ of harmonic 1-forms.

On the set $\mathrm{Harm}^1(M,g)$ of harmonic 1-forms, we put the following "Euclidean" norm: If $H \in \mathrm{Harm}^1(M,g)$ and $H = \sum \lambda_i h_i$, define $|H| =: \sum |\lambda_i|$.

Given $X \in \mathrm{sym}(M,\omega)$, we consider the Hodge decomposition of $i_X\omega$ [9]: there is a unique harmonic 1-form H_X and a unique function u_X such that

$$i_X\omega = H_X + du_X.$$

Now we define a norm $\|\cdot\|$ on the space $\mathrm{symp}(M,\omega)$ by:

$$\|X\| = |H_X| + \mathrm{osc}(u_X). \tag{1}$$

For each symplectic isotopy ϕ_t, consider the Hodge decomposition of $i_{(\dot{\phi}_t)}\omega$

$$i_{(\dot{\phi}_t)}\omega = H_t^\Phi + du_t^\Phi$$

where H_t^Φ is a harmonic 1-form.

If $\Phi = (\phi_t)$ and $\Psi = (\psi_t)$ are symplectic isotopies, we define a distance:

$$D_0(\Phi,\Psi) = \int_0^1 \|\dot{\phi}_t - \dot{\psi}_t\|\,\mathrm{d}t.$$

Denote by $\Phi^{-1} = (\phi_t^{-1})$ and by $\Psi^{-1} = (\psi_t^{-1})$ the inverse isotopies.

Consider the following more symmetrical distance:

$$D(\Phi,\Psi) = \left(D_0(\Phi,\Psi) + D_0\left(\Phi^{-1},\Psi^{-1}\right)\right)/2.$$

We define the *symplectic distance* on $\mathrm{Iso}(M,\omega)$ by:

$$d_{\mathrm{symp}}(\Phi,\Psi) = \bar{d}(\Phi,\Psi) + D(\Phi,\Psi).$$

Definition 3.1. The *symplectic topology* τ on $\mathrm{Iso}(M,\omega)$ is the metric topology defined by the distance d_{symp}.

The distance d_{sym} reduces to the Hamiltonian distance d_{ham} when Φ and Ψ are Hamiltonian isotopies and the "symplectic topology" reduces to the "Hamiltonian topology" of [7] on paths in $\mathrm{Ham}(M,\omega)$.

Definition 3.2. A homeomorphism h is said to be a strong symplectic homeomorphism if there exists a continuous path $\lambda : [0,1] \to \mathrm{Homeo}(M)$ such that $\lambda(0) = \mathrm{Id}$, $\lambda(1) = h$ and a Cauchy sequence $\Phi^n \in \mathrm{Iso}(M,\omega)$ (for the d_{symp} distance), which converges to λ in the C^0 topology (induced by the norm $\bar{d}$). We will denote by $\mathrm{SSympeo}(M,\omega)$ the set of all strong symplectic homeomorphisms.

Our main results are:

Theorem 3.3. *The symplectic topology τ on* $\mathrm{Iso}(M,\omega)$ *is natural* (*independent of all choices involved in its definition*).

Theorem 3.4. *Let (M,ω) be a closed symplectic manifold. Then* $\mathrm{SSympeo}(M,\omega)$ *is an arcwise connected topological group, containing* $\mathrm{Hameo}(M,\omega)$ *as a normal subgroup, and contained in the identity component* $\mathrm{Sympeo}_0(M,\omega)$ *of* $\mathrm{Sympeo}(M,\omega)$. *If M is simply connected,* $\mathrm{SSympeo}(M,\omega)=\mathrm{Hameo}(M,\omega)$. *Finally, the commutator subgroup* $[\mathrm{SSympeo}(M,\omega),\mathrm{SSympeo}(M,\omega)]$ *of* $\mathrm{SSympeo}(M,\omega)$ *is contained in* $\mathrm{Hameo}(M,\omega)$.

Conjecture 3.5. *The inclusion* $\mathrm{SSympeo}(M,\omega)\subset\mathrm{Sympeo}_0(M,\omega)$ *is strict.*

This conjecture implies that the identity component $\mathrm{Homeo}^{\Omega}(S^2)_0$ in the group of measure preserving homeomorphisms of S^2 is not a simple group (a conjecture of Oh–Muller [7]).

4. A Hofer-like metric on $\mathrm{Symp}(M,\omega)_0$

The length $l(\Phi)$ of a symplectic isotopy $\Phi=(\phi_t)$ is the distance $D(\Phi,\mathbf{1})$ where $\mathbf{1}$ is the identity isotopy.

The Hofer norm $\|\phi\|_H$ of a Hamiltonian diffeomorphism ϕ is

$$\|\phi\|_H=\inf\bigl(l(\Phi)\bigr), \tag{2}$$

where the infimum is taken over all Hamiltonian isotopies Φ from ϕ to the identity, and the Hofer distance d_H between ϕ and ψ is defined as $d_H(\phi,\psi)=\|\phi\psi^{-1}\|_H$.

It is easy to see that d_H is a bi-invariant pseudo-distance. The proof that it is non-degenerate is very involved: it was given by Hofer [4] for $\mathbb{R}^{2n}$ and by Lalonde–McDuff in full generality [6].

We announce the following result:

Theorem 4.1. *Let (M,ω) be a compact symplectic manifold. For any $\phi\in\mathrm{Symp}(M,\omega)$, let*:

$$e(\phi)=\inf\bigl(l(\Phi)\bigr),$$

where the infimum is taken over all symplectic isotopies Φ from ϕ to the identity. The mapping $d:\mathrm{Symp}(M,\omega)_0\times\mathrm{Symp}(M,\omega)_0\to\mathbb{R}\cup\{\infty\}$, $(\phi,\psi)\mapsto d(\phi,\psi)=e(\phi\psi^{-1})$ *is a (non bi-invariant) distance on the identity component* $\mathrm{Symp}(M,\omega)_0$ *in the group* $\mathrm{Symp}(M,\omega)$. *Moreover, the metric topology defined by d is natural (independent of the choices made to define it).*

Note that, if $\phi\in\mathrm{Ham}(M,\omega)$, we have $e(\phi)\leqslant\|\phi\|_H$.

5. Sketch of the proofs

The detailed proofs will appear in [2] and [3]. If $X\in\mathrm{symp}(M,\omega)$ and $\|X\|$ is the norm in (1) defined using a basis $B=(h_1,\dots,h_k)$ of harmonic 1-forms for some riemannian metric g, one shows that if g' is another riemannian metric, the g'-harmonic components (h_i') of (h_i) form a basis of g'-harmonic 1-forms and $\|X\|$ equals to the norm in (1) obtained using g' and (h_i'). Since all basis of the finite dimensional vector space of harmonic 1-forms are equivalent, all the norms given by (1) are equivalent.

The proof of Theorem 3.4 consists mainly into showing that if $\Phi_n,\Psi_n\in\mathrm{Iso}(M,\omega)$ are Cauchy sequences, then $\Phi_n\cdot\Psi_n$ and $(\Phi)_n^{-1}$ are Cauchy. This is tedious.

In the proof of Theorem 4.1, only the non-degeneracy is delicate. To prove it, we decompose a symplectic isotopy $\Phi=(\phi_t)$ into a product $\phi_t=\sigma_t\cdot h_t$ of a "harmonic" isotopy σ_t and a Hamiltonian isotopy h_t. If $l(\Phi)$ is arbitrarily small, then the Hofer length of h_t is also small. We show, using Ono's theorem that the Calabi group is discrete [8], that σ_1 is a Hamiltonian diffeomorphism. Then we use a "canonical" deformation of σ_t (see [1]), Proposition II.3.1), to construct a Hamiltonian isotopy from σ_1 to the identity, and show that the Hofer length of the resulting isotopy can be controlled and kept small. This proves that $\phi=\phi_1$ has a Hofer norm arbitrarily small; hence it is the identity.

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