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Differential Topology

Versal braid monodromy

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Abstract

We extend the method of Zariski to determine the braid monodromy group of the discriminant of a versal unfolding of a hypersurface singularity from low-dimensional generic subunfoldings to highly non-generic ones. At the expense of an induction over adjacent singularities, it is thus possible to neglect genericity issues and perturb by very simple polynomials only. ***To cite this article: M. Lönne, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Monodromie verselle des tresses. Nous étendons la méthode de Zariski, qui sert à déterminer la monodromie des tresses pour le discriminant d'une déformation verselle de singularités d'une hypersurface, d'une sous-déformation générique de basse dimension vers des déformations hautement non génériques. Aux frais d'une induction sur les singularités adjacentes, mais sans devoir prendre en compte les questions de généricité, il est possible ainsi de déformer par des polynômes très simples. ***Pour citer cet article : M. Lönne, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

1. Introduction

Due to a wealth of remarkable properties the versal unfolding of a hypersurface singularity f has been a challenge for a long time. It governs the Milnor fibre, the intersection lattice and the monodromy of f. These data are also obtained from a Morsification or inductively from singularities adjacent to f, cf. [1,3].

We are interested in the braid monodromy group of discriminant complements of versal unfoldings. This invariant of f is given also by discriminant complements of suitable subunfoldings, which we call Zariskification, since their existence is due to Zariski's result on generic hypersurface sections.

In this Note we define the notion of *versal braid monodromy group* for any subunfolding, which takes into account the braid monodromies of versal unfoldings of adjacent singularities, and we show that it coincides with the braid monodromy group of f under a weak transversality condition, cf. [2].

We thus gain much flexibility in the choice of subunfoldings, which has been exploited with great success in the computation [6,5] of the braid monodromy and the fundamental group of discriminant complements for Brieskorn–Pham polynomials. (In this Note we generally employ the notions and notations of [1].)

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2. Distinguished classes of unfoldings

It is well known that a *Morsification* of an isolated singularity $f \in \mathcal{O}_n$ can be understood as an unfolding

$$f_{t,u} : \mathbf{C}^n \times \mathbf{C}^2, \quad 0 \to \mathbf{C}, \quad x, t, u \mapsto f_{t,u}(x) = f_t(x) - u, \quad f_0(x) = f(x),$$

of hypersurfaces, induced by a map $\hat{\varphi} : \mathbf{C}^2 \to \mathbf{C}^\mu \cong \mathcal{O}_n / J_f$ to the base of a miniversal unfolding with image of $\hat{\varphi}$ restricted to constant $t_0 \neq 0$ transversal to the discriminant $\mathcal{D}$, i.e. transversal to $\mathcal{D}_{\text{reg}}$ and disjoint from $\mathcal{D}_{\text{sing}}$.

In analogy we want to define a *Zariskification* to be an unfolding of functions

$$f_{s,t} : \mathbf{C}^n \times \mathbf{C}^2, \quad 0 \to \mathbf{C}, \quad x, s, t \mapsto f_{s,t}(x), \quad f_{0,0}(x) = f(x), \quad f_{s,t}(0) = 0, \quad \frac{\partial f}{\partial t}(0) \neq 0,$$

which is induced by a map $\varphi : \mathbf{C}^2, 0 \to \mathbf{C}^{\mu - 1} \cong \mathsf{m}_n / J_f$ to the base of a truncated miniversal unfolding such that the image of φ restricted to constant $s_0 \neq 0$ is transversal to the function bifurcation set $\mathcal{B}$.

There is also an analogue for the fact that Morsifications determine the monodromy group of f:

Lemma 2.1. *The braid monodromy group of a Zariskification is equal to the braid monodromy group.*

Proof. It suffices to recall the analogous argument for Morsifications. The essential input is, that a curve $\operatorname{im} \varphi|_{s = s_0}$, $s_0 \neq 0$ is transversal to $\mathcal{B}$ and therefore the induced map on fundamental groups surjects. $\square$

Next we define *versal braid monodromy groups* for unfoldings $F : \mathbf{C}^n \times \mathbf{C}^k \to \mathbf{C}$, which are *$\mathcal{B}$-transversal* in the sense that the inducing map φ_F does not map all tangents at 0 to the Zariski tangent space of $\mathcal{B}$. For sake of clarity we restrict ourselves to *tame* unfoldings, where outside codimension two, each function is tame, i.e. critical values may only coincide for non-degenerate critical points.

Given a family of functions f_λ with f_0 tame and f_λ Morse for $\lambda \neq 0$ the associated family p_λ of discriminant polynomials consists of monic, univariate polynomials which only have simple zeroes for $\lambda \neq 0$:

$$p_\lambda(u) = 0 \quad \Leftrightarrow \quad \exists x : \operatorname{grad}_x f_\lambda(x) = 0, \quad f_\lambda(x) = u.$$

Let v_j denote the roots of p_0. Then for $\varepsilon > 0$ and $0 < \delta \ll \varepsilon$ sufficiently small, the discriminant complement $Y = \mathbf{C} \times D_\delta \setminus p_\lambda^{-1}(0)$ is trivializable over the disc D_δ in the complement of $\bigcup_j B_\varepsilon(v_j)$, Fig. 1.

In any fibre Y_λ, $0 < |\lambda| < \delta$ we assign a group of mapping classes choosing generators – for each v_j – supported on punctured discs $D_j = Y_\lambda \cap B_\varepsilon(v_j)$:

In case that v_j is a multiple root of p_0, which is the image of a single critical point c_j of f, we assign the braid monodromy group for the germ of f at c_j consisting of mapping classes supported on D_j.

In case v_j is the image of non-degenerate critical points of f, we choose the group of mapping classes of D_j which fix the punctures and thus correspond to pure braids.

Given a tame $\mathcal{B}$-transversal unfolding we assign groups of mapping classes to each tame function using local slices to the bifurcation locus. The *versal braid monodromy group* is then defined to be generated by all such classes

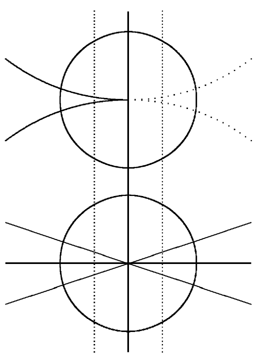


Fig. 1. Polynomial family.

transported along all possible paths to a reference fibre. It is determined as a subgroup of the braid group Br_n up to conjugacy.

We have a result simplifying the computation of versal braid monodromy groups:

Proposition 2.2. *If a family of polynomials is induced from a $\mathcal{B}$-transversal unfolding and induces a surjection of fundamental groups of bifurcation complements, the versal braid monodromy groups are isomorphic.*

Proof. Choose compatible base points. Given a generator β in the versal braid monodromy there exists by definition a disc slicing a component B of the bifurcation locus and a path γ in the bifurcation complement such that β is obtained by transport along γ from a mapping class ϕ associated to the family over the disc. By surjectivity there is another disc transversal to B and a path γ' which both lift to the induced family. Moreover there is an arc α arbitrarily close to B_{reg}. Stratified isotopy shows that the versal braid monodromies of the discs are identified under transport along α. Thus there is a braid β' obtained from ϕ by transport along α and γ' which is in the versal braid monodromy of the induced family. Since β and β' are equal up to transport along the composite of α with both paths, also β is in the monodromy of the induced family by the surjectivity property again. □

3. Comparison of braid monodromies

Theorem 3.1. *The braid monodromy group of a function f is equal to the versal braid monodromy group of any of its $\mathcal{B}$-transversal unfoldings which is tame.*

Proof. Given a $\mathcal{B}$-transversal unfolding we may induce another one meeting the hypotheses of Proposition 2.2

$$f_{s',t}: \mathbf{C}^n \times \mathbf{C}^2, \quad 0 \to \mathbf{C}, \quad f_{0,0}(x) = f(x), \quad f_{s',t}(0) = 0, \quad \frac{\partial f}{\partial t}(0) \notin T_{\mathrm{Zar}}\mathcal{B},$$

which thus has the same versal braid monodromy. By the condition on the tangent the unfolding $f_{0,t} + sg$ is a Zariskification for generic $g \in \mathrm{m}_n$. Hence for all sufficiently small $\varepsilon \neq 0$ the two-parameter family $F = f_{\varepsilon,t} + sg$ has the same braid monodromy as any versal unfolding of f while the one-parameter family $F|_{s=0} = f_{\varepsilon,t}$ has the same versal braid monodromy as the given $\mathcal{B}$-transversal unfolding.

For each point y_i in the bifurcation set on the line $s = 0$ let U_i be a small ball in the base of F centered at y_i. We fix a sufficiently small tubular neighbourhood N_η of the line $s = 0$, such that the bifurcation set of $F|_{N_\eta}$ is in the union of the U_i with singular locus in a subset of the y_i. The braid monodromy of $F|_{N_\eta}$ is thus equal to the braid monodromy of f. On the other hand it is generated by the braid monodromies of the $F|_{U_i}$ and parallel transport over the complement of the U_i. This should be compared to the fact that the versal braid monodromy of $F|_{s=0}$ is generated by the versal braid monodromies of $F|_{E_i}$ – where E_i denotes the intersection of $s = 0$ with U_i – and parallel transport over the complement of the E_i.

So it remains to prove that the versal braid monodromy of $F|_{E_i}$ is equal to the braid monodromy of $F|_{U_i}$ for each i, since the complement of the U_i in N_η retracts onto the complement of the E_i on $s = 0$.

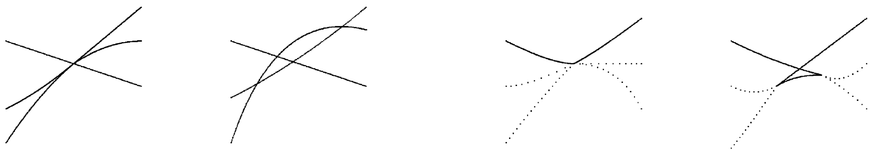
Let us thus consider a single ball U and the discriminant family of $F|_U$. Its restriction to E is a discriminant family with a single singular fibre and has a local description as in Section 2 which extends for U sufficiently small. The complement Y of the discriminant in $\mathbf{C} \times U$ is trivializable over U in the complement of balls $B_\varepsilon(v_j)$ centered at the roots v_j on the fibre over y.

The braid monodromy of $F|_U$ and the versal braid monodromy of $F|_E$ can thus be considered as a group of mapping classes which are supported on the intersection $\bigcup_j D_j$ of a local Milnor fibre with $\bigcup_j B_\varepsilon(v_j)$.

According to the decomposition of the discriminant into connected components $\mathcal{D}_j$ over U, the bifurcation locus decomposes, $\mathcal{B} = \bigcup_j \mathcal{B}_j$, where each $\mathcal{B}_j$ is the branch locus of the finite map of $\mathcal{D}_j$ onto U. Since the $B_\varepsilon(v_j)$ are disjoint, the braid monodromy transformation along a simple geometric element based at the chosen Milnor fibre and associated to $\mathcal{B}_j$ can be chosen with support in D_j.

Consider first a root v_j which is the value of non-degenerate critical points of the function F_y. Its local discriminant $\mathcal{D}_j$ in $B_\varepsilon(v_j)$ has smooth branches in bijection to the preimages. Hence all mapping classes in the braid monodromy of $F|_U$ restrict to mapping classes of D_j which fix the punctures pointwise.

On the other hand $E' := U \cap \{s = \eta\}$ is transversal to the bifurcation set, so the divisorial discriminant components in $B_\varepsilon(v_j)$ meet pairwise, transversally, and over distinct points of the bifurcation set $\mathcal{B}_j \cap E'$. This implies that the braid



(a) smooth branches (b) irreducible branch

Fig. 2. Local Zariskifications.

monodromy of $F|_{E'}$ contains all pure mapping classes of D_j, i.e. the group of mapping classes which are supported on D_j and fix the punctures pointwise. Hence this braid monodromy contains all mapping classes we assign to v_j to get the versal braid monodromy group of $F|_E$.

Similarly we argue in case the root v_j is the value of a unique critical point c_j. Then $B_\varepsilon(v_j)$ can be considered as a discriminant family induced from the base of a versal truncated unfolding of the function at c_j. It is in fact a Zariskification, see Fig. 2, since its bifurcation set $\mathcal{B}_j$ is met by E in a single point only and transversally by E'. Hence the braid monodromy of $F|_U$ contains the braid monodromy of the function at c_j considered as mapping classes on D_j extended by the identity to the Milnor fibre of $F|_U$, which is just what we assigned to v_j to get the versal braid monodromy group. □

Note that with a straightforward generalisation of versal braid monodromy to the non-tame case this claim holds in general. To demonstrate its power we show how our method is exploited in the case of a Brieskorn–Pham polynomial, cf. [5]. By the following proposition the inductive regress is through Brieskorn–Pham polynomials of decreasing codimension only which ends at polynomials of type A_k, and we may work with linear unfoldings only, considered in [4], where explicit formulae are known for the discriminant:

Proposition 3.2. *The braid monodromy group of a Brieskorn Pham singularity given by $f(x) = \sum_i x_i^{l_i}$ is generated by the versal braid monodromy groups of the families*

$$f_\lambda: \quad x \mapsto f(x) - \lambda x_1 - \sum_{i>1} \varepsilon_i x_i, \qquad g_\alpha|_{|\alpha|\leqslant 1}: \quad x \mapsto f(x) - x_1 - \alpha \sum_{i>1} \varepsilon_i x_i,$$

where $0 < \varepsilon_2, \ldots, \varepsilon_n \ll 1$ are positive real constants such that both families are tame.

[The tameness condition is open: At degenerate critical points the Hessian vanishes, which happens only for $\lambda = 0$ and $\alpha = 0$. In the first case the critical points and critical values of f_0 coincide with those of $f_0|_{x_1=0}$, so they are in bijection if the latter is a Morse function, which is an open condition. In the analogous second case it suffices to see that g_0 restricted to the x_1-axis is the Morse function $x_1^{l_1} - x_1$.]

Proof. By Theorem 3.1 the versal braid monodromy of the unfolding $\lambda, \alpha, u \mapsto f - \lambda x_1 - \alpha \sum_{i>1} \varepsilon_i x_i$ is equal to the braid monodromy of f. So it suffices to show that Proposition 2.2 applies.

Since the bifurcation set in the λ, α parameter plane is quasi-homogeneous and contains both axes, we may deduce by the method of Zariski and van Kampen that the fundamental group of the complement is generated by paths in a line parallel to the λ-axis and a path which is geometric for the λ-axis. Since such paths lie in the base of the family f_λ respectively $g_\alpha|$ we are done. □

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