

Number Theory

On period spaces for p -divisible groups

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Abstract

In their book Rapoport and Zink constructed rigid analytic period spaces for Fontaine's filtered isocrystals, and period morphisms from moduli spaces of p -divisible groups to some of these period spaces. We determine the image of these period morphisms, thereby contributing to a question of Grothendieck. We give examples showing that only in rare cases the image is all of the Rapoport–Zink period space. **To cite this article:** U. Hartl, C. R. Acad. Sci. Paris, Ser. I 346 (2008).

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Résumé

Espaces de périodes pour les groupes p -divisible. Dans leur livre, Rapoport et Zink ont construit des espaces de périodes, rigides analytiques pour les isocristaux filtrés de Fontaine. Ils ont construits également des morphismes de périodes entre des espaces modulaires des groupes de Barsotti–Tate et certains de leurs espaces de périodes. Dans cette Note nous déterminons l'image des morphismes de périodes, contribuant ainsi à une question de Grothendieck. Nous donnons des exemples montrant que l'image ne coïncide que rarement avec tout l'espace de périodes de Rapoport–Zink. **Pour citer cet article :** U. Hartl, C. R. Acad. Sci. Paris, Ser. I 346 (2008).

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On fixe un groupe de Barsotti–Tate \mathbb{X} sur $\mathbb{F}_p^{\text{alg}}$ de hauteur h et dimension d . On pose $W := W(\mathbb{F}_p^{\text{alg}})$, l'anneau des vecteurs de Witt, et $K_0 := W[\frac{1}{p}]$. Soit (D, φ_D) le F -isocristal covariant de \mathbb{X} , $\mathcal{F} := \text{Grass}(h - d; D)$ la grassmannienne des sous-espaces de dimension $h - d$ de D , et \mathcal{F}^{an} l'espace K_0 -analytique associé au sens de Berkovich [3,4]. Un point $L_K \in \mathcal{F}^{\text{an}}$ a valeurs dans une extension complète value de rang un K de caractéristique 0 est considéré comme un K -sous-espace $L_K \subset D_K := D \otimes_{K_0} K$. D'après Rapoport et Zink [19, 1.18], L_K est appelé *faiblement admissible* si l'isocristal filtré $(D, \varphi_D, F^\bullet D_K)$ avec $F^{-1}D_K = D_K$, $F^0D_K = L_K$, $F^1D_K = (0)$ est faiblement admissible au sens de Fontaine [11]. L'ensemble,

$$\mathcal{F}_{wa}^{\text{an}} := \{L_K \in \mathcal{F}^{\text{an}} : L_K \text{ est faiblement admissible}\} \subset \mathcal{F}^{\text{an}}$$

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est un sous-espace K_0 -analytique ouvert d’après [19, Proposition 1.36] et [14, Proposition 1.3]. C’est un exemple d’espaces p -*adic de périodes*, que Rapoport et Zink ont construit dans [19, Proposition 1.36] en toute généralité.

De même, ils ont construit un espace modulaire formel \mathcal{G} sur W pour les paires (X, ρ) formée des groupes de Barsotti–Tate X sur des schémas S sur W , dans lesquels p est localement nilpotent, et des quasi-isogénies $\rho : \mathbb{X}_{\bar{S}} \rightarrow X_{\bar{S}}$ sur $\bar{S} := V(p) \subset S$; voir le Théorème 1.3. Soit \mathcal{G}^{an} l’espace K_0 -analytique associé à \mathcal{G} . La théorie de Grothendieck–Messing [18] associe à l’objet universel (X, ρ) sur \mathcal{G} une extension,

$$0 \rightarrow (\text{Lie } X^\vee)_{\mathcal{G}^{\text{an}}}^\vee \rightarrow \mathbb{D}(X_{\bar{\mathcal{G}}})_{\mathcal{G}^{\text{an}}} \rightarrow \text{Lie } X_{\mathcal{G}^{\text{an}}} \rightarrow 0$$

des faisceaux localement libres sur \mathcal{G}^{an} . Par la nature cristalline de $\mathbb{D}(\cdot)$, la quasi-isogénie $\rho : \mathbb{X}_{\bar{\mathcal{G}}} \rightarrow X_{\bar{\mathcal{G}}}$ induit un isomorphisme $\mathbb{D}(\rho)_{\mathcal{G}^{\text{an}}} : \mathbb{D}(\mathbb{X}_{\bar{\mathcal{G}}})_{\mathcal{G}^{\text{an}}} \xrightarrow{\sim} \mathbb{D}(X_{\bar{\mathcal{G}}})_{\mathcal{G}^{\text{an}}}$ et l’image inverse $\mathbb{D}(\rho^{-1})_{\mathcal{G}^{\text{an}}}(\text{Lie } X^\vee)_{\mathcal{G}^{\text{an}}}^\vee$ définit un point de \mathcal{F}^{an} à valeurs dans \mathcal{G}^{an} . Le morphisme induit $\pi^{\text{an}} : \mathcal{G}^{\text{an}} \rightarrow \mathcal{F}^{\text{an}}$ est appelé *morphisme de périodes*. Il se factorise par $\mathcal{F}_{wa}^{\text{an}}$ dû à [19, 5.27].

Dans cette Note on détermine l’image de π^{an} de la manière suivante. Dans la Proposition 3.1 on associe à chaque point $L = L_K \in \mathcal{F}^{\text{an}}$ à valeurs dans K un φ -module \mathcal{M}_L sur l’anneau $\tilde{\mathbf{B}}_{\text{rig}}^\dagger$ (« la clôture algébrique de l’anneau de Robba »; voir §2) de K . On montre dans le Théorème 3.3 que l’ensemble,

$$\mathcal{F}_a^{\text{an}} := \{L \in \mathcal{F}^{\text{an}} : \mathcal{M}_L \text{ est isocline de pente zero}\}$$

est un espace K_0 -analytique ouvert de $\mathcal{F}_{wa}^{\text{an}}$. En général, l’inclusion $\mathcal{F}_a^{\text{an}} \subset \mathcal{F}_{wa}^{\text{an}}$ est stricte. On donne un exemple de ce fait dans 3.6 et on montre dans 3.5 que $\mathcal{F}_a^{\text{an}}$ est l’image du morphisme de périodes $\pi^{\text{an}} : \mathcal{G}^{\text{an}} \rightarrow \mathcal{F}^{\text{an}}$.

1. A question of Grothendieck

Fix a Barsotti–Tate group \mathbb{X} over $\mathbb{F}_p^{\text{alg}}$ of height h and dimension d . Let $W := W(\mathbb{F}_p^{\text{alg}})$ be the ring of Witt vectors and let $K_0 := W[\frac{1}{p}]$. Let \mathcal{O}_K be a complete discrete valuation ring with residue field $\mathbb{F}_p^{\text{alg}}$ and fraction field K of characteristic 0. To every Barsotti–Tate group X over \mathcal{O}_K with $\mathbb{X} \cong X \otimes_{\mathcal{O}_K} \mathbb{F}_p^{\text{alg}} =: X_{\mathbb{F}_p^{\text{alg}}}$ the theory of Grothendieck–Messing [18] associates an extension

$$0 \longrightarrow (\text{Lie } X^\vee)_K^\vee \longrightarrow \mathbb{D}(\mathbb{X})_K \longrightarrow \text{Lie } X_K \longrightarrow 0$$

where $\mathbb{D}(\mathbb{X})_K$ is the crystal of Grothendieck–Messing evaluated on K . We denote by \mathcal{F} the Grassmannian of $(h - d)$ -dimensional subspaces of $\mathbb{D}(\mathbb{X})_{K_0}$.

Problem 1.1 (A. Grothendieck [12, Remarque 3, p. 435]). *Describe the subset of \mathcal{F} formed by the points $(\text{Lie } X^\vee)_K^\vee$ where X is any deformation of \mathbb{X} over any complete discrete valuation ring \mathcal{O}_K with residue field $\mathbb{F}_p^{\text{alg}}$ and fraction field K of characteristic 0.*

This problem is, as yet, unsolved. However, a contribution was made by Rapoport and Zink [19] in the following way. Set $D := \mathbb{D}(\mathbb{X})_{K_0}$ and $\varphi_D := \mathbb{D}(\text{Frob}_{\mathbb{X}})_{K_0}$. Let $L_K \in \mathcal{F}$ be a point defined over a complete, rank one valued extension K of K_0 (which not necessarily is discrete). View L_K as a K -subspace of $D_K := D \otimes_{K_0} K$. One defines the *Newton slope* $t_N(D, \varphi_D, L_K) := \text{ord}_p(\det \varphi_D)$ and the *Hodge slope* $t_H(D, \varphi_D, L_K) := \dim_K L_K - \dim_K D_K$. Following Fontaine [11] and Rapoport–Zink [19, 1.18], the point $L_K \in \mathcal{F}$ is called *weakly admissible* if

$$t_N(D, \varphi_D, L_K) = t_H(D, \varphi_D, L_K) = -d \quad \text{and} \quad t_N(D', \varphi_D|_{D'}, L_K \cap D'_K) \geq t_H(D', \varphi_D|_{D'}, L_K \cap D'_K)$$

for all φ_D -stable K_0 -subspaces $D' \subset D$. Let \mathcal{F}^{rig} be the rigid analytic space, and \mathcal{F}^{an} the K_0 -analytic space in the sense of Berkovich [3,4] associated with \mathcal{F} .

Theorem 1.2 (Rapoport–Zink [19, Proposition 1.36]). *The set $\mathcal{F}_{wa}^{\text{rig}} := \{L_K \in \mathcal{F}^{\text{rig}} : L_K \text{ is weakly admissible}\}$ is an open rigid analytic subspace.*

The space $\mathcal{F}_{wa}^{\text{rig}}$ is an example for the p -*adic period domains* constructed more generally in [19, Proposition 1.36] for arbitrary filtered isocrystals. The proof of Rapoport and Zink even shows that $\mathcal{F}_{wa}^{\text{rig}}$ is the rigid analytic space

associated to an open K_0 -analytic subspace $\mathcal{F}_{wa}^{an} \subset \mathcal{F}^{an}$; see [14, Proposition 1.3]. The period domain \mathcal{F}_{wa}^{rig} contains the set of Grothendieck’s Problem 1.1. To explain this let $\mathcal{N}ilp_W$ be the category of W -schemes on which p is locally nilpotent. For an $S \in \mathcal{N}ilp_W$ we set $\bar{S} := V(p) \subset S$.

Theorem 1.3 (Rapoport–Zink [19, Theorem 2.16]). *The functor $\mathcal{G} : \mathcal{N}ilp_W \rightarrow Sets$*

$$S \longmapsto \left\{ \text{isomorphism classes of pairs } (X, \rho) \text{ where } X \text{ is a Barsotti–Tate group over } S \text{ and } \rho : \mathbb{X}_{\bar{S}} \rightarrow X_{\bar{S}} \text{ is a quasi-isogeny} \right\}$$

is pro-representable by a formal scheme locally formally of finite type over W .

To \mathcal{G} one can associate a rigid analytic space \mathcal{G}^{rig} by Berthelot’s construction [5], and also a K_0 -analytic space \mathcal{G}^{an} . Rapoport and Zink [19, 5.16] construct a period morphism $\check{\pi}_1^{rig} : \mathcal{G}^{rig} \rightarrow \mathcal{F}_{wa}^{rig}$ as follows. By the theory of Grothendieck–Messing [18], the universal Barsotti–Tate group X over \mathcal{G} gives rise to an extension

$$0 \longrightarrow (\text{Lie } X^\vee)_{\mathcal{G}^{rig}}^\vee \longrightarrow \mathbb{D}(X_{\bar{\mathcal{G}}})_{\mathcal{G}^{rig}} \longrightarrow \text{Lie } X_{\mathcal{G}^{rig}} \longrightarrow 0$$

of locally free sheaves on \mathcal{G}^{rig} . The quasi-isogeny $\rho : \mathbb{X}_{\bar{\mathcal{G}}} \rightarrow X_{\bar{\mathcal{G}}}$ induces by the crystalline nature of $\mathbb{D}(\cdot)$ an isomorphism $\mathbb{D}(\rho)_{\mathcal{G}^{rig}} : \mathbb{D}(\mathbb{X})_{\mathcal{G}^{rig}} \xrightarrow{\sim} \mathbb{D}(X_{\bar{\mathcal{G}}})_{\mathcal{G}^{rig}}$ and the preimage $\mathbb{D}(\rho^{-1})_{\mathcal{G}^{rig}} (\text{Lie } X^\vee)_{\mathcal{G}^{rig}}^\vee$ defines a \mathcal{G}^{rig} -valued point of \mathcal{F}^{rig} . By [19, 5.27] the induced morphism $\mathcal{G}^{rig} \rightarrow \mathcal{F}^{rig}$ factors through \mathcal{F}_{wa}^{rig} . This is the period morphism $\check{\pi}_1^{rig}$. The same construction also gives a K_0 -analytic period morphism $\check{\pi}_1^{an} : \mathcal{G}^{an} \rightarrow \mathcal{F}^{an}$.

Theorem 1.4 (Rapoport–Zink, Colmez–Fontaine, Breuil, Kisin). *The set of Problem 1.1 is contained in the subset \mathcal{F}_{wa}^{rig} . The latter equals the image of the period morphism $\check{\pi}_1^{rig} : \mathcal{G}^{rig} \rightarrow \mathcal{F}^{rig}$ in the sense of sets.*

Proof. Let $L_K = (\text{Lie } X^\vee)_K^\vee$ be a point in Grothendieck’s set, given by a Barsotti–Tate group X over \mathcal{O}_K with $\mathbb{X} \cong X_{\mathbb{F}_p^{alg}}$. Over $\mathcal{O}_K/(p)$ this isomorphism lifts by rigidity [9, Appendix] to a quasi-isogeny $\rho : \mathbb{X}_{\mathcal{O}_K/(p)} \rightarrow X_{\mathcal{O}_K/(p)}$ and (X, ρ) gives a point of $\mathcal{G}(\mathcal{O}_K)$. By construction $\check{\pi}_1^{rig}(X, \rho)_K = L_K$. So the point L_K belongs to the image of $\check{\pi}_1^{rig}$, which in turn lies in \mathcal{F}_{wa}^{rig} . It remains to show that every K -valued point $L_K \in \mathcal{F}_{wa}^{rig}$ for K/K_0 finite lies in the image of $\check{\pi}_1^{rig}$. By the theorem of Colmez–Fontaine [8] the filtered isocrystal (D, φ_D, L_K) is admissible, i.e. arises from a crystalline p -adic Galois representation V . By Breuil [6, Theorem 1.4] there is a Barsotti–Tate group X over \mathcal{O}_K and an isomorphism $T_p X_K \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong V$ if $p > 2$. Kisin [17, Corollary 2.2.6] extended Breuil’s theorem to $p = 2$ and reproved the Colmez–Fontaine Theorem. Fontaine’s functor D_{cris} transforms the isomorphism $T_p X_K \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong V$ into an isomorphism $\bar{\rho}_* : (D, \varphi_D, L_K) \xrightarrow{\sim} (\mathbb{D}(X)_{K_0}, \mathbb{D}(\text{Frob}_X)_{K_0}, (\text{Lie } X^\vee)_K^\vee)$ of filtered isocrystals. This defines a quasi-isogeny $\bar{\rho} : \mathbb{X} \rightarrow X_{\mathbb{F}_p^{alg}}$ which again by rigidity lifts to a quasi-isogeny $\rho : \mathbb{X}_{\mathcal{O}_K/(p)} \rightarrow X_{\mathcal{O}_K/(p)}$. So L_K lies in the image of the period morphism. \square

Unfortunately an open rigid analytic subspace of \mathcal{F}^{rig} , like the image of $\check{\pi}_1^{rig}$, is not uniquely determined by its underlying set of points. This however is true for K_0 -analytic spaces. Therefore it is natural to ask: What is the image of $\check{\pi}_1^{an}$? Examples of Genestier–Lafforgue and the author (see Example 3.6 below) show that this image is in general smaller than \mathcal{F}_{wa}^{an} . As [19, 1.37 and 5.53] indicate, Rapoport and Zink were aware of this phenomenon. We determine the image of $\check{\pi}_1^{an}$ in Section 3. But before, we need to recall some of the rings of Fontaine Theory.

2. Some of Fontaine’s rings

Let K/K_0 be a complete, rank one valued field extension, let C be the completion of an algebraic closure of K , and let \mathcal{O}_C be its valuation ring. Starting out from C various rings are defined in Fontaine Theory. For details on their construction see Colmez [7]. We follow his notation.

We let $\tilde{\mathbf{E}}^+ := \{u = (u^{(n)})_{n \in \mathbb{N}_0} : u^{(n)} \in \mathcal{O}_C, (u^{(n+1)})^p = u^{(n)}\}$. With the multiplication $uv := (u^{(n)}v^{(n)})_{n \in \mathbb{N}_0}$, the addition $u + v := (\lim_{m \rightarrow \infty} (u^{(m+n)} + v^{(m+n)})p^m)_{n \in \mathbb{N}_0}$, and the valuation $v_{\mathbf{E}}(u) := v_C(u^{(0)})$, $\tilde{\mathbf{E}}^+$ becomes a complete

valuation ring of rank one with algebraically closed fraction field, called $\tilde{\mathbf{E}}$, of characteristic p . Fix primitive p^n th roots of unity $\varepsilon^{(n)}$ such that $\varepsilon := (1, \varepsilon^{(1)}, \varepsilon^{(2)}, \dots)$ is an element of $\tilde{\mathbf{E}}^+$.

Let $\tilde{\mathbf{A}} := W(\tilde{\mathbf{E}})$ and consider the automorphism $\varphi = W(\text{Frob}_p)$ of $\tilde{\mathbf{A}}$. For an element $u \in \tilde{\mathbf{E}}$ let $[u] \in \tilde{\mathbf{A}}$ be its Teichmüller representative.

If $x = \sum_{i=0}^\infty p^i [x_i] \in \tilde{\mathbf{A}}$ then we set $w_k(x) := \min\{v_{\mathbf{E}}(x_i) : i \leq k\}$.

For $r > 0$ let $\tilde{\mathbf{A}}^{(0,r]} := \{x \in \tilde{\mathbf{A}} : \lim_{k \rightarrow +\infty} w_k(x) + \frac{k}{r} = +\infty\}$ and let $\tilde{\mathbf{B}}^{(0,r]} := \tilde{\mathbf{A}}^{(0,r]}[\frac{1}{p}]$.

On $\tilde{\mathbf{B}}^{(0,r]}$ there is a valuation defined for $x = \sum_{i \gg -\infty} p^i [x_i]$ as $v^{(0,r]}(x) := \min\{w_k(x) + \frac{k}{r} : k \in \mathbb{Z}\} = \min\{v_{\mathbf{E}}(x_i) + \frac{i}{r} : i \in \mathbb{Z}\}$.

Let $\tilde{\mathbf{B}}^{[0,r]}$ be the Fréchet completion of $\tilde{\mathbf{B}}^{(0,r]}$ with respect to the family of semi-valuations $v^{[s,r]}(x) := \min\{v^{(0,s]}(x), v^{(0,r]}(x)\}$ for $0 < s \leq r$. The logarithm $t := \log[\varepsilon]$ converges to an element in $\tilde{\mathbf{B}}^{[0,1]}$.

Let $\tilde{\mathbf{B}}_{\text{rig}}^\dagger := \bigcup_{r>0} \tilde{\mathbf{B}}^{[0,r]}$. The homomorphism φ gives rise to a bicontinuous isomorphism $\varphi : \tilde{\mathbf{B}}^{[0,r]} \xrightarrow{\sim} \tilde{\mathbf{B}}^{[0,r/p]}$ and thus induces an automorphism of $\tilde{\mathbf{B}}_{\text{rig}}^\dagger$. It satisfies $\varphi(t) = pt$.

Finally there is a homomorphism $\theta : \tilde{\mathbf{B}}^{(0,1]} \rightarrow C$ sending $\sum_{i \gg -\infty} p^i [x_i]$ to $\sum_{i \gg -\infty} p^i x_i^{(0)}$ which extends by continuity to $\tilde{\mathbf{B}}^{[0,1]}$. The element t lies in the kernel of θ .

Definition 2.1. A φ -module over $\tilde{\mathbf{B}}_{\text{rig}}^\dagger$ is a finite free $\tilde{\mathbf{B}}_{\text{rig}}^\dagger$ -module \mathcal{M} together with a φ -linear automorphism $\varphi_{\mathcal{M}} : \mathcal{M} \rightarrow \mathcal{M}$.

The following structure theorem was proved by Kedlaya [16, Theorem 4.5.7].

Theorem 2.2. Every φ -module \mathcal{M} over $\tilde{\mathbf{B}}_{\text{rig}}^\dagger$ is isomorphic to $\bigoplus_i \mathcal{M}_{c_i, d_i}$ where $\mathcal{M}_{c,d} = (\tilde{\mathbf{B}}_{\text{rig}}^\dagger)^{\oplus d}$,

$$\varphi_{\mathcal{M}_{c,d}} = \begin{pmatrix} 0 & & & p^c \\ & \ddots & & \\ & & 1 & \\ & & & 0 \end{pmatrix} \cdot \varphi \quad \text{for } c, d \in \mathbb{Z} \text{ with } (c, d) = 1 \text{ and } d > 0.$$

If one requires $\frac{c_i}{d_i} \leq \frac{c_{i+1}}{d_{i+1}}$ for all i , then the integers c_i, d_i are uniquely determined by \mathcal{M} .

One defines the *degree* of \mathcal{M} as $\text{deg } \mathcal{M} := \sum_i c_i$.

3. The construction of $\mathcal{F}_a^{\text{an}}$

As above let $D = \mathbb{D}(\mathbb{X})_{K_0}$ and $\varphi_D = \mathbb{D}(\text{Frob}_{\mathbb{X}})_{K_0}$. Let $(\mathcal{D}^{[0,1]}, \varphi_{\mathcal{D}}) := (D, \varphi_D) \otimes_{K_0} \tilde{\mathbf{B}}^{[0,1]}$ and consider the morphism $1 \otimes \theta : \mathcal{D}^{[0,1]} \rightarrow D \otimes_{K_0} C$. By a variant of a construction of Berger [1, §II] every point $L = L_K \in \mathcal{F}^{\text{an}}$, with values in a field K as in Section 2, defines a φ -module over $\tilde{\mathbf{B}}_{\text{rig}}^\dagger$ as follows.

Proposition 3.1. (See [14, Proposition 4.1].) There exists a uniquely determined $\tilde{\mathbf{B}}^{[0,1]}$ -submodule $t\mathcal{D}^{[0,1]} \subset \mathcal{M}_L^{[0,1]} \subset \mathcal{D}^{[0,1]}$ such that $(1 \otimes \theta)(\mathcal{M}_L^{[0,1]}) = L_K \otimes_K C$ and $\varphi_{\mathcal{D}} : \mathcal{M}_L^{[0,1]} \xrightarrow{\sim} \mathcal{M}_L^{[0,1]} \otimes_{\tilde{\mathbf{B}}^{[0,1]}} \tilde{\mathbf{B}}^{[0,1/p]}$ is an isomorphism. In particular $\mathcal{M}_L^{[0,1]}$ defines a φ -module $\mathcal{M}_L := \mathcal{M}_L^{[0,1]} \otimes_{\tilde{\mathbf{B}}^{[0,1]}} \tilde{\mathbf{B}}_{\text{rig}}^\dagger$ over $\tilde{\mathbf{B}}_{\text{rig}}^\dagger$.

The following results are proved in [14].

Theorem 3.2. (See [14, Theorem 4.4].) $\text{deg } \mathcal{M}_L = t_N(D, \varphi_D, L_K) - t_H(D, \varphi_D, L_K)$.

Consider the subset $\mathcal{F}_a^{\text{an}} := \{L \in \mathcal{F}^{\text{an}} : \mathcal{M}_L \cong \mathcal{M}_{0,1}^{\oplus h}\}$ of the K_0 -analytic space \mathcal{F}^{an} .

Theorem 3.3. (See [14, Theorem 5.2].) The set $\mathcal{F}_a^{\text{an}}$ is an open K_0 -analytic subspace of $\mathcal{F}_{\text{wa}}^{\text{an}}$.

Remark 3.4. (On the proof of Theorem 3.3. See also Remark 3.7 below.) The inclusion $\mathcal{F}_a^{\text{an}} \subset \mathcal{F}_{wa}^{\text{an}}$ is seen as follows. If $D' \subset D$ is a φ_D -stable K_0 -subspace then $(D', \varphi_D|_{D'}, L_K \cap D_K)$ defines by Proposition 3.1 a φ -submodule $\mathcal{M}'_L \subset \mathcal{M}_L$. Since $\mathcal{M}_{0,1}^{\oplus h}$ is “semistable” of slope zero we conclude by [16, Lemma 3.4.8] that

$$\frac{t_N(D', \varphi_D|_{D'}, L_K \cap D'_K) - t_H(D', \varphi_D|_{D'}, L_K \cap D'_K)}{\dim_{K_0} D'} = \frac{\deg \mathcal{M}'_L}{\dim_{K_0} D'} \geq \frac{\deg \mathcal{M}_L}{\dim_{K_0} D} = 0$$

with equality if $D' = D$. Hence $\mathcal{F}_a^{\text{an}} \subset \mathcal{F}_{wa}^{\text{an}}$. On the other hand Berger’s proof [1] of the Colmez–Fontaine Theorem shows that this inclusion induces a bijection on rigid analytic points (with K/K_0 finite), or more generally points of $\mathcal{F}_{wa}^{\text{an}}$ with values in a discretely valued field K .

Theorem 3.5. $\mathcal{F}_a^{\text{an}}$ is the image of the period morphism $\tilde{\pi}_1^{\text{an}} : \mathcal{G}^{\text{an}} \rightarrow \mathcal{F}_{wa}^{\text{an}}$.

Proof. Let $L_K \in \mathcal{F}^{\text{an}}$ be a K -valued point in the image of $\tilde{\pi}_1^{\text{an}}$ with K/K_0 not necessarily finite. So $L_K = \mathbb{D}(\rho^{-1})_K(\text{Lie } X^\vee)_K^\vee$ for a Barsotti–Tate group X over \mathcal{O}_K and a quasi-isogeny $\rho : \mathbb{X}_{\mathcal{O}_K/(p)} \rightarrow X_{\mathcal{O}_K/(p)}$. Then the Tate module $T_p X_K$ of X induces an injection $\mathcal{M}_{0,1}^{\oplus h} \cong T_p X_K \otimes_{\mathbb{Z}_p} \tilde{\mathbf{B}}_{\text{rig}}^\dagger \hookrightarrow \mathcal{M}_L$ which must be an isomorphism by reasons of degree. This was proved in [14, Proposition 6.1]. Thus $L_K \in \mathcal{F}_a^{\text{an}}$.

Conversely let $L_K \in \mathcal{F}_a^{\text{an}}$. Then the morphism $(\tilde{\mathbf{B}}_{\text{rig}}^\dagger)^{\oplus h} \xrightarrow{\sim} \mathcal{M}_{L_K} \hookrightarrow D \otimes_{K_0} \tilde{\mathbf{B}}_{\text{rig}}^\dagger$ is represented, with respect to a K_0 -basis of D , by a matrix $M \in M_h(\tilde{\mathbf{B}}_{\text{rig}}^\dagger)$ with $tM^{-1} \in M_h(\tilde{\mathbf{B}}_{\text{rig}}^\dagger)$. Then in fact $M, tM^{-1} \in M_h(\tilde{\mathbf{B}}_{\text{rig}}^\dagger) \subset M_h(\mathbf{B}_{\text{cris}}^+)$ by [2, Proposition I.4.1]. For notation see [2, §I.1] or [14, §2]. So M defines an isomorphism $\mathbf{B}_{\text{cris}}^{\oplus h} \xrightarrow{\sim} D \otimes_{K_0} \mathbf{B}_{\text{cris}}$ compatible with Frobenius, which maps $(\mathbf{B}_{\text{cris}}^+)^{\oplus h}$ onto the preimage of $L_K \otimes_K C$ under the map $\text{id} \otimes \theta : D \otimes_{K_0} \mathbf{B}_{\text{cris}}^+ \rightarrow D \otimes_{K_0} C$. This means that $(D, \varphi_D, L_K)^\vee$ is admissible in the sense of Faltings’ [10, Definition 1]. Note that Faltings uses contravariant Dieudonné modules. By [10, Theorems 9 and 14], $L_K = \mathbb{D}(\rho^{-1})_K(\text{Lie } X^\vee)_K^\vee$ for a Barsotti–Tate group X over \mathcal{O}_K and a quasi-isogeny $\rho : \mathbb{X}_{\mathcal{O}_K/(p)} \rightarrow X_{\mathcal{O}_K/(p)}$, hence L_K lies in the image of $\tilde{\pi}_1^{\text{an}}$. \square

There are many examples showing that only in rare cases $\mathcal{F}_a^{\text{an}} = \mathcal{F}_{wa}^{\text{an}}$. We mention one here. Similar examples are due to A. Genestier and V. Lafforgue.

Example 3.6. Let $D = K_0^{\oplus 5}$ and

$$\varphi_D = \begin{pmatrix} 0 & & & & \\ & p^{-3} & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 0 \end{pmatrix} \cdot \varphi.$$

Then $h = 5, d = 3$ and $\mathcal{F} = \text{Grass}(2, 5)$. Since the isocrystal (D, φ_D) is simple $\mathcal{F}_{wa}^{\text{an}} = \mathcal{F}^{\text{an}}$. Let $L = L_K \in \mathcal{F}^{\text{an}}$. Then

$$\mathcal{M}_{-3,5} = \mathcal{D} \supset \mathcal{M}_L \supset t\mathcal{D} \cong \mathcal{M}_{2,5}.$$

By Theorem 2.2, $\mathcal{M}_L \cong \bigoplus_i \mathcal{M}_{c_i, d_i}$ with $\sum_i d_i = \text{rk } \mathcal{M}_L = 5$ and $\sum_i c_i = \deg \mathcal{M}_L = 0$. Moreover by [16, Lemma 3.4.8] all the weights c_i/d_i must lie between $-3/5$ and $2/5$. So either $\mathcal{M}_L \cong \mathcal{M}_{0,1}^{\oplus 5}$ or $\mathcal{M}_L \cong \mathcal{M}_{-1,2} \oplus \mathcal{M}_{1,3}$.

Now one easily checks that

$$\begin{aligned} \text{Hom}_\varphi(\mathcal{M}_{-1,2}, \mathcal{D}) &= \text{Hom}_\varphi(\mathcal{M}_{-1,2}, \mathcal{M}_{-3,5}) \\ &= \left\{ A = \begin{pmatrix} \varphi^5(x) & x \\ \varphi^{11}(x) & \varphi^6(x) \\ \varphi^{17}(x) & \varphi^{12}(x) \\ \varphi^{23}(x) & \varphi^{18}(x) \\ \varphi^{29}(x) & \varphi^{24}(x) \end{pmatrix} : x = \sum_{v \in \mathbb{Z}} p^v \varphi^{-10v}([u]), u \in \mathbf{E}, 0 < v_{\mathbf{E}}(u) < \infty \right\}. \end{aligned}$$

The bad situation $\mathcal{M}_L \cong \mathcal{M}_{-1,2} \oplus \mathcal{M}_{1,3}$ occurs if and only if L_K is generated by the columns of such a matrix $\theta(A) \in C^{5 \times 2}$, since then the homomorphism A factors through \mathcal{M}_L and this forbids $\mathcal{M}_L \cong \mathcal{M}_{0,1}^{\oplus 5}$ by [16, Lemma 3.4.8]. Since obviously such L_K exist, this proves that the inclusion $\mathcal{F}_a^{\text{an}} \subset \mathcal{F}_{wa}^{\text{an}}$ is strict.

Remark 3.7. (On the proof of Theorem 3.3.) One can explain the idea for the proof of Theorem 3.3 by means of this example. Namely the complement $\mathcal{F}^{\text{an}} \setminus \mathcal{F}_a^{\text{an}}$ is the image of the continuous map from the compact set $\{u \in \tilde{\mathbf{E}}^+ : 1 \leq v_{\mathbf{E}}(u) \leq p^{10}\}$ given by $u \mapsto \theta(A)$.

Finally we would like to mention that the ideas for the results presented in this article were inspired by our analogous theory in equal characteristic [13], see also our dictionary [15].

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