Available online at www.sciencedirect.com

ScienceDirect

C. R. Acad. Sci. Paris, Ser. I 346 (2008) 849–852

COMPTES RENDUS MATHEMATIQUE

http://france.elsevier.com/direct/CRASS1/

Partial Differential Equations

Analytic singularities for long range Schrödinger equations

André Martinez [a,1], Shu Nakamura [b], Vania Sordoni [a]

[a] *Università di Bologna, Dipartimento di Matematica, Piazza di Porta San Donato 5, 40127 Bologna, Italy*
[b] *Graduate School of Mathematical Science, University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo, Japan 153-8914*

Received 18 June 2008; accepted 17 July 2008

Available online 8 August 2008

Presented by Jean-Michel Bony

Abstract

We consider the Schrödinger equation associated to long range perturbations of the flat Euclidean metric (in particular, potentials growing subquadratically at infinity are allowed). We construct a modified quantum free evolution $G_0(s)$ acting on Sjöstrand's spaces, and we characterize the analytic wave front set of the solution $e^{-itH}u_0$ of the Schrödinger equation, in terms of the semiclassical exponential decay of $G_0(-th^{-1})\mathbf{T}u_0$, where $\mathbf{T}$ stands for the Bargmann-transform. The result is valid for $t<0$ near the forward non-trapping points, and for $t>0$ near the backward non-trapping points. ***To cite this article: A. Martinez et al., C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Singularités analytiques pour des équations de Schrödinger à longue portée. On considère l'équation de Schrödinger associée à des perturbations à longue portée de la métrique euclidienne plate (en particulier, on autorise des potentiels qui croissent de manière sub-quadratique à l'infini). On construit une évolution quantique modifiée $G_0(s)$ agissant sur des espaces de Sjöstrand, et on caractérise le front d'onde analytique de la solution $e^{-itH}u_0$ de l'équation de Schrödinger en termes de décroissance exponentielle semiclassique de $G_0(-th^{-1})\mathbf{T}u_0$, où $\mathbf{T}$ désigne la tranformation de Bargmann. Le résultat est valable pour $t<0$ près des points non captifs dans l'avenir, et pour $t>0$ près des points non captifs dans le passé. ***Pour citer cet article : A. Martinez et al., C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

1. Notations and main result

We are concerned with the analytic wave front set of the solution $u = e^{-itH}u_0$ of the Schrödinger equation,

$$\text{(Sch):}\quad \begin{cases} i\frac{\partial u}{\partial t} = Hu; \\ u_{|t=0} = u_0, \end{cases}$$

E-mail addresses: martinez@dm.unibo.it (A. Martinez), shu@ms.u-tokyo.ac.jp (S. Nakamura), sordoni@dm.unibo.it (V. Sordoni).

[1] Partly supported by Università di Bologna, Funds for Selected Research Topics and Founds for Agreements with Foreign Universities.

1631-073X/$ – see front matter
doi:10.1016/j.crma.2008.07.010

with $u_0 \in L^2(\mathbb{R}^n)$, and H of the form,

$$H = \frac{1}{2}\sum_{j,k=1}^{n} D_j a_{j,k}(x) D_k + \frac{1}{2}\sum_{j=1}^{n}\big(a_j(x)D_j + D_j a_j(x)\big) + a_0(x)$$

where $D_j = -\mathrm{i}\partial_{x_j}$, and the coefficients $a_\alpha(x)$ satisfy to the following assumptions. For $\nu > 0$ we denote

$$\Gamma_\nu = \big\{z \in \mathbb{C}^n \mid |\operatorname{Im} z| < \nu\langle \operatorname{Re} z\rangle\big\}.$$

Assumption A. For each α, $a_\alpha(x) \in C^\infty(\mathbb{R}^n)$ is real-valued and can be extended to a holomorphic function on Γ_ν with some $\nu > 0$. Moreover, for $x \in \mathbb{R}^n$, the matrix $(a_{j,k}(x))_{1\leqslant j,k\leqslant n}$ is symmetric and positive definite, and there exists $\sigma \in (0, 1]$ such that,

$$\big|a_{j,k}(x) - \delta_{j,k}\big| \leqslant C_0\langle x\rangle^{-\sigma}, \quad j,k = 1,\dots,n,$$
$$\big|a_j(x)\big| \leqslant C_0\langle x\rangle^{1-\sigma}, \quad j = 1,\dots,n,$$
$$\big|a_0(x)\big| \leqslant C_0\langle x\rangle^{2-\sigma},$$

for $x \in \Gamma_\nu$ and with some constant $C_0 > 0$.

In particular, H is essentially selfadjoint on $C_0^\infty(\mathbb{R}^n)$, and we use the same letter H for its unique selfadjoint extension on $L^2(\mathbb{R}^n)$.

We denote by $p(x,\xi) := \frac{1}{2}\sum_{j,k=1}^{n} a_{j,k}(x)\xi_j\xi_k$ the principal symbol of H, and by $H_0 := -\frac{1}{2}\Delta$ the free Laplace operator. For any $(x,\xi) \in \mathbb{R}^{2n}$, we also denote by $(y(t;x,\xi), \eta(t;x,\xi)) = \exp tH_p(x,\xi)$ the Hamilton flow associated with p, and we say that a point $(x_0,\xi_0) \in T^*\mathbb{R}^n\backslash 0$ is forward non-trapping when $|y(t,x_0,\xi_0)| \to \infty$ as $t \to +\infty$. In that case, there exists $\xi_+(x_0,\xi_0) \in \mathbb{R}^n$, such that,

$$\big|\xi_+(x_0,\xi_0) - \eta(t,x_0,\xi_0)\big| \to 0 \quad \text{as } t \to +\infty.$$

We also introduce the Bargmann-FBI transform $\mathbf{T}$ defined by,

$$\mathbf{T}u(z,h) = \int \mathrm{e}^{-(z-y)^2/2h} u(y)\,\mathrm{d}y$$

(where $z \in \mathbb{C}^n$ and $h > 0$ is a small extra-parameter), and we recall from [12] that a point $(x,\xi) \in T^*\mathbb{R}^n\backslash 0$ is not in $WF_a(u)$ if and only if there exists some $\delta > 0$ such that $\mathbf{T}u = \mathcal{O}(\mathrm{e}^{(\Phi_0(z)-\delta)/h})$ uniformly for z close enough to $x - i\xi$ and $h > 0$ small enough, where $\Phi_0(z) := (\operatorname{Im} z)^2/2$ (see also [2]). In this case, we will write: $\mathbf{T}u \sim 0$ in $H_{\Phi_0, x-i\xi}$.

Finally, we set,

$$q(x,\xi;h) := \frac{1}{2}\sum_{j,k=1}^{n} a_{j,k}(x)\xi_j\xi_k + h\sum_{j=1}^{n} a_j(x)\xi_j + h^2 a_0(x),$$

and we denote by $(\tilde{x}(t,x,\xi;h), \tilde{\xi}(t,x,\xi;h)) := \exp tH_q(x,\xi)$ its Hamilton flow.

Our main result is:

Theorem 1.1. *Suppose Assumption* A. *Then, for any $\delta_0 > 0$, there exists an h-dependent analytic function $W_h = W_h(s,\xi)$ on $[0,+\infty) \times \{|\xi| > \delta_0\}$ solution of,*

$$\frac{\partial W_h}{\partial s} = q(\partial_\xi W_h, \xi; h), \tag{1}$$

and such that, if one sets $\widetilde{W}_h := W_h - W_{h|_{s=0}}$, then, for any forward non-trapping point (x_0,ξ_0) with $|\xi_+(x_0,\xi_0)| > \delta_0$, and for all $T > 0$, the quantity

$$\tilde{x}(T/h, x_0, \xi_0) - \partial_\xi \widetilde{W}_h\big(T/h, \tilde{\xi}(T/h, x_0,\xi_0); h\big) \tag{2}$$

admits a limit $x_+(x_0,\xi_0)$ independent of T as $h \to 0_+$. Moreover, for any $t < 0$ and any $u_0 \in L^2(\mathbb{R}^n)$, one has the equivalence,

$$(x_0,\xi_0)\notin WF_a\big(\mathrm{e}^{-\mathrm{i}tH}u_0\big) \quad\Longleftrightarrow\quad \mathrm{e}^{\mathrm{i}\widetilde{W}_h(-th^{-1},hD_z)/h}\mathbf{T}u_0\sim 0 \quad \textit{in } H_{\Phi_0,z_+(x_0,\xi_0)},$$

where $z_+(x_0,\xi_0):=x_+(x_0,\xi_0)-i\xi_+(x_0,\xi_0)$.

Remark 1.2. Since $WF_a(u)$ is conical with respect to ξ and $\xi_+(x_0,\lambda\xi_0)=\lambda\xi_+(x_0,\xi_0)$ for all $\lambda>0$, the condition $|\xi_+(x_0,\xi_0)|>\delta_0$ is not restrictive.

Remark 1.3. The operator $\mathrm{e}^{\mathrm{i}\widetilde{W}_h(-th^{-1},hD_z)/h}$ appearing in the statement is not defined by the Spectral Theorem, but rather as a Fourier integral operator acting on Sjöstrand's spaces.

Remark 1.4. Actually, Eq. (1) needs not be satisfied by W_h, and the result remains valid with any W_h such that (2) admits a limit, and $\frac{\partial W_h}{\partial s}-q(\partial_\xi W_h,\xi;h)=\mathcal{O}(\langle s\rangle^{-1-\sigma})$ uniformly for $s=\mathcal{O}(h^{-1})$, $h\to 0_+$.

Remark 1.5. In the short-range case $\sigma>1$, one can actually take $W(s,\xi)=s\xi^2/2$, so that $\mathrm{e}^{\mathrm{i}W(-th^{-1},hD_z)/h}$ just becomes $\mathrm{e}^{-\mathrm{i}tD_z^2/2}$, and the function $\mathrm{e}^{\mathrm{i}W(-th^{-1},hD_z)/h}\mathbf{T}u_0$ coincides with $\mathbf{T}(\mathrm{e}^{-\mathrm{i}tH_0}u_0)$. Thus, in that case, one recovers the result of [4].

Remark 1.6. Of course, there is a similar result for (x_0,ξ_0) backward non-trapping and $t>0$.

Remark 1.7. Our result is an extension to the analytic category of those of [8], and, as for [3,4], it has been mainly motivated by a whole series of results, both in the C^∞ and in the analytic categories, obtained during these very last few years [1,6–11,13]. A much longer list of references on this problem, with results going back to the 80's, can be found in [3], and the details of the proof can be found in [5].

2. Sketch of proof

2.1. Preliminaries

Replacing u_0 by $\mathrm{e}^{\mathrm{i}tH}u_0$ and changing t to $-t$, we can reformulate the result as follows: For any $t>0$, one has the equivalence,

$$(x_0,\xi_0)\notin WF_a(u_0) \quad\Longleftrightarrow\quad \mathrm{e}^{\mathrm{i}\widetilde{W}(th^{-1},hD_z)/h}\mathbf{T}\big(\mathrm{e}^{-\mathrm{i}tH}u_0\big)\sim 0 \quad \text{in } H_{\Phi_0,z_+(x_0,\xi_0)}.$$

Changing the time scale, we set $\tilde{v}(s):=\mathbf{T}\mathrm{e}^{-\mathrm{i}hsH}u_0$, and, by a standard result of Sjöstrand's theory (see [12] Proposition 7.4 and [4] Section 4), we see that $\tilde{v}(s)$ is solution of,

$$\mathrm{i}h\frac{\partial\tilde{v}}{\partial s}\sim\widetilde{Q}\tilde{v}(s) \quad \text{in } H^{\mathrm{loc}}_{\Phi_0}, \tag{3}$$

where $\widetilde{Q}$ is an analytic pseudodifferential operator in the sense of [12], with symbol $\tilde{q}$ verifying,

$$\tilde{q}(z,\zeta;h)=q(z+\mathrm{i}\zeta,\zeta;h)+\mathcal{O}\big(h\langle z\rangle^{-1-\sigma}+h^2\langle z\rangle^{-\sigma}+h^3\langle z\rangle^{1-\sigma}\big)$$

uniformly as $|\mathrm{Re}\,z|\to\infty$, $h\to 0_+$, $|\mathrm{Im}\,z|+|\zeta|=\mathcal{O}(1)$.

2.2. The modified free evolution

At first, we choose $R\geqslant 1$ sufficiently large in order to make the map,

$$J_s:\xi\mapsto\tilde{\xi}\big(s,R\xi/|\xi|,\xi;h\big)$$

a global diffeomorphism from $\{|\xi|>\delta\}$ to its image, with $\delta>0$ such that $J_s(|\xi|>\delta)\supset\{|\xi|>\delta_0\}$. Then, we define the function W_h by the formula,

$$W_h(s,\xi):=R|\xi|+\int_0^s q\big(\tilde{x}\big(s,R\xi/|\xi|,J_s^{-1}(\xi)\big),\xi;h\big)\,\mathrm{d}s',$$

and, by standard Hamilton–Jacobi theory, we see that W_h solves (1), and that one has $\partial_\xi W_h(s,\xi) = \tilde{x}(s, R\xi/|\xi|, J_s^{-1}(\xi))$. Using arguments taken from [8], we also see that the quantity (2) admits a limit $x_+(x_0,\xi_0)$ independent of T as $h \to 0_+$.

Finally, the Fourier integral operator $e^{i\widetilde{W}_h(s,hD_z)/h}$ is defined by

$$e^{i\widetilde{W}_h(s,hD_z)/h} v(z;h) := \frac{1}{(2\pi h)^n} \int\limits_{\gamma(s,z)} e^{i(z-y)\eta/h + i\widetilde{W}_h(s,\eta)/h} v(y)\,dy\,d\eta, \tag{4}$$

where $\gamma(s,z)$ is a convenient contour of $\mathbb{C}^{2n}$ (it is a good contour in the sense of [12], with some uniformity as $s \to +\infty$).

2.3. Completion of the proof

Conjugating the equation with $e^{i\widetilde{W}_h(s,hD_z)/h}$, we obtain the new equation,

$$ih\partial_s w(s) = L(s)w(s)$$

where $w(s) := e^{i\widetilde{W}_h(s,hD_z)/h}\tilde{v}(s)$, and the symbol of the analytic pseudodifferential operator $L(s)$ verifies,

$$\ell(s,z,\zeta;h) = q\big(z + i\zeta + \partial_\zeta \widetilde{W}_h(s,\zeta), \zeta; h\big) - (\partial_s W_h)(s,\zeta) + \mathcal{O}\big(h\langle s\rangle^{-1-\sigma}\big),$$

locally uniformly in (z,ζ), and uniformly for $s \in [0, T/h]$ with $T > 0$ fixed. In particular, using (1), we see that $\ell = \mathcal{O}(\langle s\rangle^{-1-\sigma})$, and $\ell = \ell_0(z + i\zeta, \zeta; h) + \mathcal{O}(h\langle s\rangle^{-1-\sigma})$, where the Hamilton flow R_s of ℓ_0 is given by,

$$R_s(x,\xi;h) := \big(\tilde{x}(s,x,\xi;h) - \partial_\xi \widetilde{W}\big(s, \tilde{\xi}(s,x,\xi;h)\big), \tilde{\xi}(x,x,\xi;h)\big).$$

Therefore, we are reduced to a short range situation, where the arguments of [4] can be performed and permit to conclude.

References

[1] A. Hassel, J. Wunsch, The Schrödinger propagator for scattering metrics, Ann. of Math. (2) 162 (1) (2005) 487–523.

[2] A. Martinez, An Introduction to Semiclassical and Microlocal Analysis, UTX Series, Springer-Verlag, New York, 2002.

[3] A. Martinez, S. Nakamura, V. Sordoni, Analytic smoothing effect for the Schrödinger equation with long-range perturbation, Comm. Pure Appl. Math. 59 (2006) 1330–1351.

[4] A. Martinez, S. Nakamura, V. Sordoni, Analytic wave front set for solutions to Schrödinger equations, Preprint, 2007.

[5] A. Martinez, S., Nakamura, V. Sordoni, Analytic wave front set for solutions to Schrödinger equations II – long range perturbations, Preprint, 2008.

[6] S. Nakamura, Propagation of the homogeneous wave front set for Schrödinger equations, Duke Math. J. 126 (2005) 349–367.

[7] S. Nakamura, Wave front set for solutions to Schrödinger equations, Preprint, 2004, J. Funct. Anal., in press.

[8] S. Nakamura, Semiclassical singularities propagation properties for the Schrödinger equations, Preprint, 2006, J. Math. Soc. Japan, in press.

[9] L. Robbiano, C. Zuily, Microlocal analytic smoothing effect for Schrödinger equation, Duke Math. J. 100 (1999) 93–129.

[10] L. Robbiano, C. Zuily, Effet régularisant microlocal analytique pour l'équation de Schrödinger: le cas des données oscillantes, Comm. Partial Differential Equations 100 (2000) 1891–1906.

[11] L. Robbiano, C. Zuily, Analytic theory for the quadratic scattering wave front set and application to the Schrödinger equation, Astérisque 283 (2002) 1–128.

[12] J. Sjöstrand, Singularités analytiques microlocales, Astérisque 95 (1982) 1–166.

[13] J. Wunsch, Propagation of singularities and growth for Schrödinger operators, Duke Math. J. 98 (1999) 137–186.