Available online at www.sciencedirect.com
ScienceDirect
C. R. Acad. Sci. Paris, Ser. I 346 (2008) 853–856
COMPTES RENDUS MATHEMATIQUE
http://france.elsevier.com/direct/CRASS1/

Functional Analysis

Smallest singular value of random matrices with independent columns

Radosław Adamczak [a,1], Olivier Guédon [b], Alexander Litvak [c], Alain Pajor [d], Nicole Tomczak-Jaegermann [c,2]

[a] *Institute of Mathematics, University of Warsaw, Banacha 2, 02-097 Warszawa, Poland*
[b] *Université Pierre-et-Marie-Curie, Paris 6, Institut de mathématiques de Jussieu, 4, place Jussieu, 75005 Paris, France*
[c] *Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, Alberta, Canada T6G 2G1*
[d] *Équipe d'analyse et mathématiques appliquées, Université Paris Est, 5, boulevard Descartes, Champs-sur-Marne, 77454 Marne-la-Vallee cedex 2, France*

Received 30 June 2008; accepted 10 July 2008

Available online 9 August 2008

Presented by Gilles Pisier

Abstract

We study the smallest singular value of a square random matrix with i.i.d. columns drawn from an isotropic symmetric log-concave distribution. We prove a deviation inequality in terms of the isotropic constant of the distribution. ***To cite this article: R. Adamczak et al., C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Sur la plus petite valeur singulière de matrices aléatoires avec des colonnes indépendantes. On étudie la plus petite valeur singulière d'une matrice carrée aléatoire dont les colonnes sont des vecteurs aléatoires i.i.d. suivant une loi à densité log-concave isotrope. On démontre une inégalité de déviation en fonction de la constante d'isotropie. ***Pour citer cet article : R. Adamczak et al., C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

The behaviour of the smallest singular value of random matrices with i.i.d. random entries attracted a lot of attention over the years. Major results were recently obtained in [5,8–10]. In asymptotic geometry one is interested in sampling vectors uniformly distributed in a convex body. In particular the entries are not necessarily independent. In this note, we study the more general case when the columns are i.i.d. random vectors with a symmetric isotropic log-concave

E-mail addresses: radamcz@mimuw.edu.pl (R. Adamczak), guedon@math.jussieu.fr (O. Guédon), alexandr@math.ualberta.ca (A. Litvak), alain.pajor@univ-mlv.fr (A. Pajor), nicole@ellpspace.math.ualberta.ca (N. Tomczak-Jaegermann).
[1] This work was done when this author held a postdoctoral position at the Department of Mathematical and Statistical Sciences, University of Alberta in Edmonton, Alberta. The position was co-sponsored by the Pacific Institute for the Mathematical Sciences.
[2] This author holds the Canada Research Chair in Geometric Analysis.

1631-073X/$ – see front matter
doi:10.1016/j.crma.2008.07.011

distribution. We prove a deviation inequality for the smallest singular value in terms of a parameter L_μ which, in the case of sampling from a convex body, corresponds to the isotropic constant of the body.

Recall that a non-negative function f on $\mathbb{R}^n$ is called log-concave if for all $x, y \in \mathbb{R}^n$ and all $\theta \in (0,1)$, $f((1-\theta)x+\theta y) \geqslant f(x)^{1-\theta} f(y)^\theta$. In this paper a symmetric probability measure μ on $\mathbb{R}^n$ is said to be log-concave if its density f is symmetric log-concave and it is called isotropic if its covariance matrix is the identity. We will also set $L_\mu = f(0)^{1/n}$. Let us observe that if μ is an isotropic probability measure uniformly distributed on a symmetric convex body K then L_μ is the so-called isotropic constant of K. If X is a random vector, distributed according to μ, we will also write $L_X = L_\mu$.

We shall use the notation $|\cdot|$ to denote the Euclidean norm of a vector or the volume or the cardinality of a set.

Theorem 1. *Let $n \geqslant 1$ and let Γ be an $n \times n$ matrix with independent columns drawn from an isotropic symmetric log-concave probability μ. For every $\varepsilon \in (0,1)$ and all $\delta \in (0,1)$ and all $M \geqslant 1$ we have*

$$\mathbb{P}\biggl(\inf_{x\in S^{n-1}} |\Gamma x| \leqslant \varepsilon \biggl(\frac{c_1}{M L_\mu}\biggr)^{1/(1-\delta)} n^{-1/2}\biggr) \leqslant \frac{C\varepsilon}{\delta} + \mathrm{e}^{-c_2 n} + \mathbb{P}\bigl(\|\Gamma\| > M\sqrt{n}\bigr), \tag{1}$$

where $c_1, c_2 > 0$ and C are absolute constants. Moreover, if $\delta \leqslant 1 - 1/(2n)$, then

$$\mathbb{P}\biggl(\inf_{x\in S^{n-1}} |\Gamma x| \leqslant \varepsilon \biggl(\frac{c_1}{M L_\mu}\biggr)^{1/(1-\delta)} n^{-1/2}\biggr) \leqslant \frac{C\varepsilon^{1/2}}{\delta} + \mathbb{P}\bigl(\|\Gamma\| > M\sqrt{n}\bigr). \tag{2}$$

Estimates for $\mathbb{P}(\|\Gamma\| > M\sqrt{n})$, when M is a power of $\log n$, can be deduced from [6] and [3].

An important case when we have more information (that follows from a result of Aubrun [1]) is that of 1-unconditional measures. Recall that a probability measure with density f is 1-unconditional if for any $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and any $(\varepsilon_1, \dots, \varepsilon_n) \in \{-1,1\}^n$, $f(x_1, \dots, x_n) = f(\varepsilon_1 x_1, \dots, \varepsilon_n x_n)$.

Corollary 2. *If a probability μ is 1-unconditional, then Γ satisfies*

$$\mathbb{P}\Bigl(\inf_{x\in S^{n-1}} |\Gamma x| \leqslant \varepsilon n^{-1/2}\Bigr) \leqslant C\varepsilon + 2\mathrm{e}^{-cn^{1/5}},$$

where C and $c > 0$ are absolute constants. Moreover, for all $\varepsilon \in (0,1)$ we have

$$\mathbb{P}\Bigl(\inf_{x\in S^{n-1}} |\Gamma x| \leqslant \varepsilon n^{-1/2}\Bigr) \leqslant C\varepsilon^{cn^{1/5}/(2(cn^{1/5}+1))}.$$

The proof of the theorem requires the study of the isotropic constant of a sum of i.i.d. random vectors in $\mathbb{R}^n$. Let $X_1, \dots, X_n$ be independent isotropic log-concave symmetric random vectors in $\mathbb{R}^n$. Let $x \in S^{n-1}$, and set

$$Z = x_1 X_1 + \cdots + x_n X_n.$$

Then it is well known that Z is also an isotropic log-concave symmetric random vector in $\mathbb{R}^n$. If $X_1, \dots, X_n$ are 1-unconditional, then so is Z. The following theorem is of independent interest.

Theorem 3. *Let $X_1, \dots, X_n$ be i.i.d. random vectors in $\mathbb{R}^n$, distributed according to a symmetric isotropic log-concave probability μ, let $x \in S^{n-1}$ and $Z = x_1 X_1 + \cdots + x_n X_n$. Then $L_Z \leqslant C L_\mu$, where C is a universal constant.*

The proof is based on the following version of a result by Gluskin and Milman [2]. Recall that K is called a star body whenever $tK \subset K$ for all $0 \leqslant t \leqslant 1$, and in such a case $\|\cdot\|_K$ denotes its Minkowski functional.

Lemma 4. *Let $f_1, \dots, f_m$ be densities of probability measures on $\mathbb{R}^n$ and let $K \subset \mathbb{R}^n$ be a star body containing the origin in its interior. Then for all $\lambda_1, \dots, \lambda_m$ we have*

$$\biggl(\int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \biggl\|\sum_{i=1}^m \lambda_i x_i\biggr\|_K^2 \prod_{i=1}^m f(x_i)\,\mathrm{d}x_i\biggr)^{1/2} \geqslant c|K|^{-1/n} \biggl(\sum_{i=1}^m \lambda_i^2 r_i^2\biggr)^{1/2}, \tag{3}$$

where $r_i^2 = \int_0^\infty |\{x\colon\ f_i(x) \geqslant t\}|^{1+2/n}\,\mathrm{d}t \geqslant \|f_i\|_\infty^{-2/n}$ and $c > 0$ is an absolute constant.

Proof of Theorem 3. Let f be the density of μ and let g be the density of Z. By Lemma 2 in [4] there exists a star-shaped body $K \subset \mathbb{R}^n$, with 0 in its interior such that

$$g(0)^{1/n}|K|^{1/n}\left(\int_{\mathbb{R}^n} \|x\|_K^2 g(x)\,\mathrm{d}x\right)^{1/2} \leqslant C,$$

for a certain universal constant C. On the other hand, by Lemma 4 we have

$$\begin{aligned}\left(\int_{\mathbb{R}^n} \|x\|_K^2 g(x)\,\mathrm{d}x\right)^{1/2} &= \left(\mathbb{E}\|Z\|_K^2\right)^{1/2} = \left(\mathbb{E}\|x_1X_1+\cdots+x_nX_n\|_K^2\right)^{1/2} \\ &\geqslant \frac{c}{|K|^{1/n}f(0)^{1/n}}\left(\sum_{i=1}^n x_i^2\right)^{1/2} = \frac{c}{|K|^{1/n}f(0)^{1/n}}.\end{aligned}$$

Putting these two inequalities together concludes the proof. □

We pass now directly to the proof of Theorem 1 and we assume that Γ and μ satisfy the assumptions described there. Similarly as in [5,8,9], the argument relies on splitting the sphere S^{n-1} into several regions. We use the following notation from [9]:

$$\begin{aligned}&\mathit{Sparse} = \mathit{Sparse}(\delta) = \left\{x \in \mathbb{R}^n\colon \left|\mathrm{supp}(x)\right| \leqslant \delta n\right\},\\ &\mathit{Comp} = \mathit{Comp}(\delta,\rho) = \left\{x \in S^{n-1}\colon \mathrm{dist}\big(x, \mathit{Sparse}(\delta)\big) \leqslant \rho\right\},\\ &\mathit{Incomp} = \mathit{Incomp}(\delta,\rho) = S^{n-1}\setminus \mathit{Comp}(\delta,\rho).\end{aligned}$$

Proposition 5. *For all $\rho, \delta, \varepsilon \in (0,1)$ we have*

$$\mathbb{P}\Big(\inf_{x\in \mathit{Incomp}(\delta,\rho)} |\Gamma x| \leqslant \rho\varepsilon n^{-1/2}\Big) \leqslant \frac{C}{\delta}\varepsilon,$$

where C is an absolute constant.

The proof of this proposition uses Lemma 3.5 of [9] which reduces the required estimate to an estimate of probability of the form $\mathbb{P}_{X_k}(|\langle X_k^*, X_k\rangle| < \varepsilon)$, for a fixed $1 \leqslant k \leqslant n$, where X_k^* is a random vector of norm 1 independent of X_k. For each fixed value of X_k^*, $\langle X_k^*, X_k\rangle$ is a one-dimensional isotropic log-concave and symmetric random variable and therefore the latter probability can be bounded above by $C\varepsilon$, where C is a universal constant. The proof is then finished by Lemma 3.5 of [9].

Proposition 6. *Let Γ be an $n \times n$ random matrix with independent columns $X_1,\ldots,X_n$, distributed according to a symmetric isotropic log-concave probability μ. Then, for any $M > 1$ and $\delta, \rho \in (0,1)$, we have*

$$\mathbb{P}\Big(\inf_{x\in \mathit{Comp}(\delta,\rho/(2M))} |\Gamma x| \leqslant \rho\sqrt{n}\ \&\ \|\Gamma\| \leqslant M\sqrt{n}\Big) \leqslant C^n L_\mu^n M^{\delta n}\rho^{(1-\delta)n},$$

where C is an absolute constant. In particular, there exist constants $c_1, c_2 > 0$ such that for every $M > 1$ and $\delta, \rho \in (0,1)$, satisfying

$$\rho \leqslant \left(\frac{c_1}{M^\delta L_\mu}\right)^{1/(1-\delta)}$$

we have

$$\mathbb{P}\Big(\inf_{x\in \mathit{Comp}(\delta,\rho/(2M))} |\Gamma x| \leqslant \rho\sqrt{n}\ \&\ \|\Gamma\| \leqslant M\sqrt{n}\Big) \leqslant \mathrm{e}^{-c_2 n}.$$

It is easy to see that for every fixed $x \in S^{n-1}$, letting $Z = \Gamma x$, we get

$$\mathbb{P}\big(|Z| \leqslant \rho\sqrt{n}\big) \leqslant C^n L_Z^n \rho^n,$$

where C is an absolute constant. Then the proof of Proposition 6 uses Theorem 3 and an ε-net argument. More sophisticated estimates for a small ball probability for random vectors distributed according to a symmetric isotropic log-concave measure were recently proved by Paouris [7].

Proof of Theorem 1. For a fixed $\delta \in (0,1)$ and $M \geqslant 1$, we apply Proposition 6 with

$$\rho = \left(\frac{c_1}{M^\delta L_\mu}\right)^{1/(1-\delta)}$$

and get

$$\mathbb{P}\left(\inf_{x \in Comp(\delta, \rho/(2M))} |\Gamma x| \leqslant \left(\frac{c_1}{M^\delta L_\mu}\right)^{1/(1-\delta)} \sqrt{n}\right) \leqslant \mathrm{e}^{-c_2 n} + \mathbb{P}\big(\|\Gamma\| > M\sqrt{n}\big).$$

Since

$$\varepsilon \left(\frac{c_1}{M L_\mu}\right)^{1/(1-\delta)} n^{-1/2} = \varepsilon M^{-1} \left(\frac{c_1}{M^\delta L_\mu}\right)^{1/(1-\delta)} n^{-1/2} \leqslant \left(\frac{c_1}{M^\delta L_\mu}\right)^{1/(1-\delta)} \sqrt{n},$$

we also have

$$\mathbb{P}\left(\inf_{x \in Comp(\delta, \rho/(2M))} |\Gamma x| \leqslant \varepsilon \left(\frac{c_1}{M L_\mu}\right)^{1/(1-\delta)} n^{-1/2}\right) \leqslant \mathrm{e}^{-c_2 n} + \mathbb{P}\big(\|\Gamma\| > M\sqrt{n}\big).$$

Now, Proposition 5 applied with $\rho/2M$ instead of ρ and 2ε instead of ε gives

$$\mathbb{P}\left(\inf_{x \in Incomp(\delta, \rho/(2M))} |\Gamma x| \leqslant \varepsilon \left(\frac{c_1}{M L_\mu}\right)^{1/(1-\delta)} n^{-1/2}\right) \leqslant \frac{C\varepsilon}{\delta}.$$

The last two inequalities combined with the fact that $S^{n-1} = Incomp(\delta, \rho/(2M)) \cup Comp(\delta, \rho/(2M))$ and union bound allow us to conclude (1).

The proof of the “moreover part” is similar. We omit further details. □

References

[1] G. Aubrun, Sampling convex bodies: a random matrix approach, Proc. Amer. Math. Soc. 135 (2007) 1293–1303 (electronic).

[2] E. Gluskin, V. Milman, Geometric probability and random cotype 2, in: GAFA, in: Lecture Notes in Math., vol. 1850, Springer, Berlin, 2004, pp. 123–138.

[3] O. Guédon, M. Rudelson, L_p-moments of random vectors via majorizing measures, Adv. Math. 208 (2007) 798–823.

[4] M. Junge, Volume estimates for log-concave densities with application to iterated convolutions, Pacific J. Math. 169 (1995) 107–133.

[5] A.E. Litvak, A. Pajor, M. Rudelson, N. Tomczak-Jaegermann, Smallest singular value of random matrices and geometry of random polytopes, Adv. Math. 195 (2005) 491–523.

[6] S. Mendelson, A. Pajor, On singular values of matrices with independent rows, Bernoulli 12 (2006) 761–773.

[7] G. Paouris, personal communication.

[8] M. Rudelson, Invertibility of random matrices: norm of the inverse, Ann. of Math. 168 (2008) 575–600.

[9] M. Rudelson, R. Vershynin, The Littlewood–Offord problem and invertibility of random matrices, Adv. Math. 218 (2008) 600–633.

[10] T. Tao, V. Vu, Inverse Littlewood–Offord theorems and the condition number of random discrete matrices, Ann. of Math., in press.