



Algebraic Geometry

On semistable vector bundles over curves

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Abstract

Let X be a geometrically irreducible smooth projective curve defined over a field k , and let E be a vector bundle on X . Then E is semistable if and only if there is a vector bundle F on X such that $H^i(X, F \otimes E) = 0$ for $i = 0, 1$. We give an explicit bound for the rank of F . The proof uses a result of Popa for the case where k is algebraically closed. **To cite this article:** I. Biswas et al., *C. R. Acad. Sci. Paris, Ser. I 346 (2008)*.

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Résumé

Sur les fibrés vectoriels semi-stables au-dessus des courbes. Soit X une courbe projective lisse géométriquement irréductible définie sur un corps k , et soit E un fibré vectoriel sur X . E est semi-stable si et seulement s'il y a un fibré vectoriel F sur X tel que $H^i(X, F \otimes E) = 0$ pour $i = 0, 1$. Nous donnons une borne explicite pour le rang de F . La preuve utilise un résultat de Popa pour le cas où k est algébriquement clos. **Pour citer cet article :** I. Biswas et al., *C. R. Acad. Sci. Paris, Ser. I 346 (2008)*.

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1. Introduction

Let k be field with algebraic closure K , and let X be a geometrically irreducible, smooth, projective curve defined over k of genus $g \geq 2$. Recall that a vector bundle E over X is called semistable if for all subbundles of positive rank $E' \subset E$ defined over k , the inequality $\mu(E') \leq \mu(E)$ holds. Here the rational number $\mu(E') := \frac{\deg(E')}{\text{rk}(E')}$ is the slope of the vector bundle E' . It is known that E is semistable if and only if the base change $E \otimes_k K \rightarrow X \times_k K$ is semistable; this is proved in [4, p. 97, Proposition 3].

Assume that there exists a second vector bundle F on X , such that $H^*(X, F \otimes E) = 0$, meaning $H^0(X, F \otimes E) = 0 = H^1(X, F \otimes E)$. We call such a vector bundle F to be *cohomologically orthogonal* to E . This condition implies that $\chi(F \otimes E) = 0$, or equivalently, $\mu(F) + \mu(E) = g - 1$. If there were a destabilizing bundle $E' \subset E$, then we would have $\mu(F \otimes E') > g - 1$ implying $h^0(F \otimes E') > 0$. This is absurd because $H^0(F \otimes E') \subset H^0(F \otimes E) = 0$. Consequently, the statement $H^*(X, F \otimes E) = 0$ implies the semistability of E (and of F as well).

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Faltings showed in [2] that for k algebraically closed, the converse is also true: if E is semistable, then there exists a vector bundle F with $H^*(X, F \otimes E) = 0$. Popa showed in [5, Theorem 5.3] that F can be chosen to have a prescribed rank and determinant that depend only on the rank and degree of E .

Faltings' result generalizes to arbitrary fields k as follows. Given the semistable vector bundle E on X defined over k , it yields a cohomologically orthogonal bundle F' defined over K . This F' is then defined over some finite extension ℓ/k . The pushforward F of F' along the morphisms $X \times_k \ell \rightarrow X$ gives us a vector bundle F defined over k , which is cohomologically orthogonal to E according to the projection formula.

For perfect fields k , a bound on the rank of F is given in [1]. The main result, namely Theorem 3.1, of [1] shows that for a perfect field k , and a semistable vector bundle E on X/k , there exists a vector bundle F of a given rank R defined over k such that $H^*(X, F \otimes E) = 0$. However, the rank R of F is huge in general, and the bound in [1] is far from being optimal.

The purpose of this note is twofold: First we remove the perfectness assumption. Secondly, we improve the bound on the rank R of the sheaf F which is cohomologically orthogonal to a semistable vector bundle E .

In [1], a point outside the divisor Θ_E was constructed using [1, Corollary 2.5] and the fact that the moduli space of S -equivalence classes of rank R semistable vector bundles on X of fixed determinant is projective. Here we use the geometry of the moduli space of rank R semistable vector bundles with fixed determinant and rank, which is known to be a unirational variety.

2. Notation

k – a field

K – its algebraic closure

X – a smooth, projective curve defined over the field k which is geometrically irreducible

g – the genus of X

ω_X – the dualizing line bundle on X

E – a vector bundle on X defined over k

r – the rank $\text{rk}(E)$ of E

d – the degree $\text{deg}(E)$ of E

h – $h := \text{gcd}(r, d)$, the greatest common divisor of r and d

$m - m := \lceil \frac{r^2+1}{8h} \rceil$ the round up

$R - R := 2rm$ (this will be the rank of a cohomologically orthogonal bundle F over k or over some finite extension L/k if k is a finite field)

$D - D := m(2r(g-1) - 2d)$ (this will be the degree of F)

$L - L := \omega_X^{\otimes mr} \otimes \det(E)^{\otimes -2m}$ (this will be the determinant of the bundle F).

3. Infinite fields

Theorem 1. *Suppose k is a infinite field, and let X be a smooth projective geometrically irreducible curve over k . For a vector bundle E of rank r and degree d over X , the following three statements are equivalent:*

- (i) *The vector bundle E is semistable.*
- (ii) *There exists a vector bundle F on X defined over k such that $H^*(X, F \otimes E) = 0$.*
- (iii) *There exists a vector bundle F on X defined over k of rank R and determinant L such that $H^*(X, F \otimes E) = 0$.*

Proof. Note that (iii) \Rightarrow (ii) is trivial, and (ii) \Leftrightarrow (i) was discussed in the introduction. We will show that (i) \Rightarrow (iii). Here are the steps of the proof.

- (1) Since the statement is twist invariant we may assume (replacing E by $E \otimes \omega_X^{\otimes n}$ for an appropriate integer n) that $2 - 3g \leq \mu(E) < -g$.
- (2) We take R and L as above, and let F be any semistable vector bundle of rank R and determinant L defined over K . We obtain $\mu(F) = g - 1 - \mu(E) > 2g - 1$. Thus, for any point $P \in X(K)$ we have $\mu(F(-P)) > 2g - 2$.

By semistability of F we conclude that $\text{Hom}(F(-P), \omega_X) = 0$. From Serre duality we have $H^1(X, F(-P)) = 0$. Therefore, it follows that any semistable bundle F of rank R and determinant L is globally generated.

- (3) Since F is globally generated we obtain a surjection $H^0(X, F) \otimes_k \mathcal{O}_X \rightarrow F$. For a general $(R + 1)$ -dimensional linear subspace

$$W \subset H^0(X \times_k K, F \otimes_k K),$$

the corresponding homomorphism $W \otimes_K \mathcal{O}_{X \times_k K} \rightarrow F \otimes_k K$ is surjective because X is smooth of dimension one.

- (4) If $\det F \cong L$, then for any surjection $\pi : \mathcal{O}_X^{\oplus(R+1)} \rightarrow F$, the kernel is L^{-1} . Thus, all those F (and a little bit more) are overparameterized by $\mathbb{P}(V)$ where $V := \text{Hom}(L^{-1}, \mathcal{O}_X^{\oplus(R+1)})^\vee$. We consider the morphisms

$$X \xleftarrow{p} X \times \mathbb{P}(V) \xrightarrow{q} \mathbb{P}(V)$$

and have the universal short exact sequence on $X \times \mathbb{P}(V)$:

$$0 \rightarrow L^{-1} \boxtimes \mathcal{O}(-1) \rightarrow p^* \mathcal{O}_X^{\oplus(R+1)} \rightarrow \mathcal{F} \rightarrow 0.$$

Obviously, both $\mathbb{P}(V)$ and \mathcal{F} are defined over k .

- (5) We tensor the above short exact sequence of sheaves with p^*E , and apply the push forward q_* to $\mathbb{P}(V)$. Let

$$\rightarrow H^1(X, E \otimes L^{-1}) \otimes \mathcal{O}_{\mathbb{P}(V)}(-1) \xrightarrow{\psi_E} H^1(X, E^{\oplus(R+1)}) \otimes \mathcal{O}_{\mathbb{P}(V)} \rightarrow R^1 q_*(p^*E \otimes \mathcal{F}) \rightarrow 0$$

be the resulting long exact sequence of sheaves on $\mathbb{P}(V)$. Since $E \otimes L^{-1}$ and $E^{\oplus(R+1)}$ are semistable vector bundles of negative degree, they have no global sections. Using the Riemann–Roch theorem, we get $h^1(X, E \otimes L^{-1}) = h^1(X, E^{\oplus(R+1)}) = rD + g - 1 - d$. The support Θ_E of $R^1 q_*(p^*E \otimes \mathcal{F})$ is therefore the vanishing locus of the section $\det(\psi_E) \in H^0(\mathcal{O}_{\mathbb{P}(V)}(rD + g - 1 - d))$. Set theoretically Θ_E describes all short exact sequences

$$0 \rightarrow L^{-1} \xrightarrow{\alpha} \mathcal{O}_X^{\oplus(R+1)} \rightarrow F_\alpha \rightarrow 0$$

such that $h^1(E \otimes F_\alpha) > 0$. This includes all F_α which are not locally free, or not semistable. Popa’s result [5, Theorem 5.3] implies that with our choices of the rank R the set $\Theta_E \subset \mathbb{P}(V)$ is a divisor, or equivalently, $\det(\psi_E) \neq 0$.

- (6) Since k has infinitely many elements the k -rational points of the divisor Θ_E are a proper subset of $\mathbb{P}(V)(k)$. See also [3, p. 4, Proposition 1.3(a)]. \square

4. Finite fields

In this section we consider a field k with q elements. We will need the additional number

$$M := \lceil \log_q(r(D + g - 1) - d) \rceil$$

along with the notation from Section 2.

Theorem 2. *Suppose k is a finite field with q elements, and let X be a smooth projective geometrically irreducible curve over k . For a vector bundle E of rank r and degree d over X , the following statements are equivalent:*

- (i) *The vector bundle E is semistable.*
- (ii) *There exists a vector bundle F on X defined over k such that $H^*(X, F \otimes E) = 0$.*
- (iii) *For every field extension ℓ/k of degree at least M , there exists a vector bundle F' of rank R defined over ℓ such that $H^*(X \times_k \ell, F' \otimes (E \otimes_k \ell)) = 0$.*
- (iv) *There exists a vector bundle F on X defined over k of rank $R \cdot M$ such that $H^*(X, F \otimes E) = 0$.*

Proof. We will show that (i) \Rightarrow (iii) \Rightarrow (iv). Note that (iv) \Rightarrow (ii) is obvious, and (ii) \Leftrightarrow (i) was discussed in the introduction.

(i) \Rightarrow (iii): We follow the proof of Theorem 1 in steps 1–5. To find a point outside the divisor $\Theta_E \subset \mathbb{P}(V)$, we pass to a field extension ℓ/k with at least $\text{deg}(\Theta_E)$ elements. Thus, any field extension of degree at least M will do by our

choice of M above. By [1, Lemma 2.2], there exists a point in $\mathbb{P}(V)(\ell)$ outside Θ_E . This point corresponds to a vector bundle F' defined over ℓ such that $H^*(X \times_k \ell, F' \otimes (E \otimes_k \ell)) = 0$.

(iii) \Rightarrow (iv): We take a finite field extension ℓ/k of degree M . Now the field extension ℓ/k is Galois with Galois group $\text{Gal} = \text{Gal}(\ell/k)$. Setting $F := \bigoplus_{\gamma \in \text{Gal}} \gamma^* F'$ we obtain a vector bundle of rank $R \cdot [\ell : k]$ which is defined over k , and $H^0(X, F \otimes E) = 0$. \square

Remark. The rank of the cohomologically orthogonal bundle F in Theorem 1 (the case of infinite fields) is independent of the genus g of our curve X . However, the number M in Theorem 2 depends on g . Thus, in the case of a finite field the rank of F depends on g .

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