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Numerical Analysis

Goal-oriented anisotropic grid adaptation

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Abstract

Venditti and Darmofal have introduced a grid adaptation strategy for estimating and reducing simulation errors in functional outputs of partial differential equations. The procedure is based on an adjoint formulation in which the estimated error in the functional can be directly related to the local residual errors of both the primal and adjoint solutions. In this note, we propose an extension of this method to the anisotropic case. The strategy proposed for grid adaptation is also compared with the anisotropic Hessian approach, based on the minimization of interpolation error. *To cite this article: G. Rogé, L. Martin, C. R. Acad. Sci. Paris, Ser. I 346 (2008).*

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Résumé

Adaptation de maillage anisotrope orientée objectif. Venditti et Darmofal ont formulé une méthode d'adaptation anisotrope pour estimer et réduire l'erreur commise sur les fonctions d'intérêt issues d'équations aux dérivées partielles. La procédure est basée sur une formulation adjointe grâce à laquelle l'erreur estimée sur la fonction est directement reliée aux résidus locaux des solutions primale et duale. Dans ce compte-rendu, nous proposons une extension de cette méthode au cas anisotrope. Une comparaison avec la méthode anisotrope utilisant la matrice hessienne de la solution et visant à minimiser l'erreur d'interpolation, est également menée. *Pour citer cet article : G. Rogé, L. Martin, C. R. Acad. Sci. Paris, Ser. I 346 (2008).* © 2008 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

1. Introduction

The aim of this Note is to find a grid adaptation method designed to produce specifically tuned grids for accurately estimating a chosen functional. The procedure developed by Venditti and Darmofal [2] uses the concept of duality, in which an equivalent dual formulation of the primal problem is exploited. The primary benefit of invoking the dual problem, in the context of error estimation, is that the error in a chosen functional can be directly related to local residual errors of the primal solution through the adjoint variables. This goal-oriented method is based on the adjoint theory developed by Pierce and Giles [1]. However, when solving flow equations, or more generally PDE¹ systems, the method of refinement should take into account the relevant directions to be more efficient. This is not the case with

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the adjoint method which is fully isotropic. On the contrary, Hessian-based adaptation is an anisotropic adaptation. It is based upon the construction of a local metric using information from the Hessian matrix of the solution to remesh in space. Since this feature-based process is not directly linked to the output of interest, it can produce a mesh with more vertices than the goal-oriented method to make similar improvements to functional accuracy. Here, we present an extension of the goal-oriented procedure to the anisotropic case, without considering the Hessian matrix as has been done by Venditti and Darmofal [2]. Moreover, we will adopt a new kind of interpolation based on using characteristics.

2. Adjoint-based correction

We give an output f(W) (such as lift or drag coefficient) that depends on the solution W of the Euler equations written as R(W) = 0 over a domain Ω in \mathcal{R} . An accurate estimate of this quantity is essential for the design of aerodynamic structures. We first consider a *n*-simplex triangulation M_H of the physical domain. The discrete Euler equations denoted by $R_H(W_H) = 0$ should be solved to determine the value of the output $f_H(W_H)$. We denote by Ψ_H the solution of the adjoint problem: $\frac{\partial R_H}{\partial W_H}^T \Psi_H = \frac{\partial f_H}{\partial W_H}^T$.

Let us note M_h (h < H) a mesh finer than M_H . The notation F_h^H refers to the interpolation of a field F_H from the coarse mesh M_H to the fine mesh M_h . We start from the adjoint-based correction technique developed by Pierce and Giles [1]: $f_h(W_h) \approx f_h(W_h^H) - R_h(W_h^H) \cdot \Psi_h^H$. The correction term $R_h(W_h^H) \cdot \Psi_h^H$ can be viewed as the local error of interpolated Euler solution $R_h(W_h^H)$ weighted by their relative influence on the output Ψ_h^H . Venditti and Darmofal [2] suggest to use the nodal components of this field as adaptation parameter.

3. Computation of the adaptation parameter

This first step consists of the computation of the adaptation parameter $R_h(W_h^H) \cdot \psi_h^H$ on the coarse grid M_H , using a fine grid M_h .

In the isotropic process, the fine grid M_h is obtained by applying an iso-P2 refinement to the coarse grid M_H . Then, we make an interpolation to determine W_h^H and ψ_h^H on M_h . The adaptation parameter $R_h(W_h^H) \cdot \psi_h^H$ is computed on M_h and interpolated to M_H to obtain $(R \cdot \psi)_H$. Using an isotropic refinement to create the fine grid does not allow us to make any distinction between directions.

The new anisotropic procedure consists in creating as many new fine meshes as directions and computing the adaptation parameter on each of them.

For the 3D case, we have to refine M_H in the direction defined by the axis x (resp. y and z) to create M_{h_x} (resp. M_{h_y} and M_{h_z}). For example, to set up the mesh M_{h_y} , we proceed to the global remeshing of M_H using the uniform metric $M_y = \text{diag}(1, 4, 1)$. This strategy leads us to $h_y \approx H/2$, keeping h_x and $h_z \approx H$.

metric $M_y = \text{diag}(1, 4, 1)$. This strategy leads us to $h_y \approx H/2$, keeping h_x and $h_z \approx H$. Again, the transfer by interpolation is done to determine W_h^H and ψ_h^H on the fine grids. Instead of classical P1-interpolation, we use interpolation along characteristics to obtain $(W_{h_x}^H, W_{h_y}^H, W_{h_z}^H)$ and interpolation along inverse characteristics to obtain $(\psi_{h_x}^H, \psi_{h_y}^H, \psi_{h_z}^H)$. Then, the adaptation parameter can be computed on the three fine meshes. We have $R_{h_x}(W_{h_x}^H) \cdot \psi_{h_x}^H$ for M_{h_x} , $R_{h_y}(W_{h_y}^H) \cdot \psi_{h_y}^H$ for M_{h_y} and $R_{h_z}(W_{h_z}^H) \cdot \psi_{h_z}^H$ for M_{h_z} . Subsequently, we transfer the three fine adaptation parameters to the coarse grid M_H and denote the resulting interpolated adaptation parameters as $((R \cdot \psi)_{H_x}, (R \cdot \psi)_{H_y}, (R \cdot \psi)_{H_z})$.

4. Selection process

During this step, certain nodes of M_H will be flagged for refinement. This set of flagged nodes is noted F_H . It is a sub-part of the set Θ_H of nodes of M_H . For isotropic (resp. anisotropic) case, a node A_H of M_H is flagged for refinement if and only if: $(R \cdot \psi)_H(A_H) > \epsilon$ (resp. $\max((R \cdot \psi)_{H_x}, (R \cdot \psi)_{H_y}, (R \cdot \psi)_{H_z}) > \epsilon)$ where ϵ is a level of accuracy fixed by the user. There are two causes for not selecting a node: either the Euler residual R is close to zero, which corresponds to areas where the solution is well captured, or the adjoint ψ is close to zero, which corresponds to areas which have little influence on the value of the output of interest.

5. Remeshing process

In the isotropic process, each element containing at least one flagged node is refined likewise along all the edges. During the new anisotropic procedure, a remeshing of selected areas is performed, for which a special metric needs

to be defined for all nodes F_H . For each node $A_H^{F_H}$ belonging to F_H , we define eight symmetric matrices $N_{F_H}^{(i,j,k)}$ with

(i, j, k) in $\{1, 2\}^3$. The matrix elements depend on $A_H^{F_H}$ but, for simplicity, it has been omitted in our notation. The generic expression is:

$$N_{A_{H}^{F_{H}}}^{(i,j,k)} = \begin{pmatrix} (R \cdot \psi)_{H_{x}} & (-1)^{i} \sqrt{|(R \cdot \psi)_{H_{x}}(R \cdot \psi)_{H_{y}}|} & (-1)^{j} \sqrt{|(R \cdot \psi)_{H_{x}}(R \cdot \psi)_{H_{z}}|} \\ (-1)^{i} \sqrt{|(R \cdot \psi)_{H_{x}}(R \cdot \psi)_{H_{y}}|} & (R \cdot \psi)_{H_{y}} & (-1)^{k} \sqrt{|(R \cdot \psi)_{H_{y}}(R \cdot \psi)_{H_{z}}|} \\ (-1)^{j} \sqrt{|(R \cdot \psi)_{H_{x}}(R \cdot \psi)_{H_{z}}|} & (-1)^{k} \sqrt{|(R \cdot \psi)_{H_{z}}|} & (R \cdot \psi)_{H_{z}} \end{pmatrix}.$$

We note $u_n^{(i,j,k)}$, with *n* in {1, 2, 3}, the three column vectors of $N_{A_H^{F_H}}^{(i,j,k)}$. These eight matrices are candidates to define

directions (u_1, u_2, u_3) used to construct the local metric around $A_H^{F_H}$. At each vertex, the main direction of the anisotropic metric is this one along which the absolute value of the directional gradient of the solution $\nabla W_H(A_H^{F_H})$ is largest. Consequently, if we have:

$$|u_{n_0}^{(i_0,j_0,k_0)} \cdot \nabla W_H(A_H^{F_H})| = \max_{n,i,j,k} (|u_n^{(i,j,k)} \cdot \nabla W_H(A_H^{F_H})|).$$

then the directions of the metric are defined by $N_{A_{H}}^{(i_{0}, j_{0}, k_{0})}$. Moreover, the directional gradient gives the eigenvalues of the new local metric.

Finally, the proposed metric for anisotropic grid adaptation is given by:

$$M_{A_{H}^{F_{H}}} = N_{A_{H}^{F_{H}}}^{(i_{0},j_{0},k_{0})} \operatorname{diag}(|u_{1}^{(i_{0},j_{0},k_{0})} \cdot \nabla W_{H}|, |u_{2}^{(i_{0},j_{0},k_{0})} \cdot \nabla W_{H}|, |u_{3}^{(i_{0},j_{0},k_{0})} \cdot \nabla W_{H}|)(N_{A_{H}^{F_{H}}}^{(i_{0},j_{0},k_{0})})^{-1}$$

Finally, we have a local metric $M_{A_H}^{F_H}$ associated with each node $A_H^{F_H}$ belonging to F_H . This metric is then used by a remeshing anisotropic software based on the Delaunay–Voronoï method.

6. Presentation of the results

The space Ω is a cube in 3D (square in 2D), uniformly refined. The state equation (R(W) = 0) is an advection problem with: $R(W) = \vec{v} \cdot \nabla W - 2\|\vec{v}\| \vec{v} \cdot \vec{X}$ where \vec{v} denotes advection velocity – defining the direction ζ with (ζ, η, τ) an orthonormal basis of \Re^3 – and \vec{X} the vector of coordinates. The function of interest f is the square of the L^2 -norm of the state on a subdomain U, with U an open subset included in Ω , i.e.: $f = \|W\|_{0,U}^2$. The exact analytical solution is: $W = \zeta^2 + g(\eta, \tau)$. Thanks to limit conditions, we assure that $g(\eta, \tau)$ is uniformly equal to zero. So the state and the observation only depend on the direction ζ .

Fig. 1 summarizes the results in the 2D and 3D case with \vec{v} equals to (1, 2) (2D case) and (1, 1, 1) (3D case). Concerning the 2D result, the figure on the left shows the analytical solution in the spatial domain of study Ω by contour plot. The black line delimits the subdomain U which impacts the function of interest f. As opposed to the goal-oriented metric (in white) which is only defined for nodes flagged during the selection process, the Hessian-metric (in red) is represented in the whole domain. As expected, our goal-oriented method finds both relevant orientation and stretching. For the 3D result, the figure on the right shows the analytical solution in the spatial domain of study Ω by isovalue surfaces. We represent the goal-oriented metric in black for the main direction and in green and red for the two other. The same conclusions can be drawn from the 3D study as from the 2D one.

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Fig. 1. 2D and 3D results.

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