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Numerical Analysis

A monotonic evaluation of lower bounds for inf-sup stability constants in the frame of reduced basis approximations

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Abstract

For accurate *a posteriori* error analysis of the reduced basis method for coercive and non-coercive problems, a critical ingredient lies in the evaluation of a lower bound for the coercivity or inf-sup constant. In this short Note, we generalize and improve the successive constraint method first presented by Huynh (2007) by providing a monotonic version of this algorithm that leads to both more stable evaluations and fewer *offline* computations. ***To cite this article: Y. Chen et al., C. R. Acad. Sci. Paris, Ser. I 346 (2008).***
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Résumé

Une évaluation monotone de brones fiables pour la constante inf-sup dans la méthode de l'approximation des bases réduites. Un ingrédient fondamental de l'analyse *a posteriori* pour l'approximation par méthode par bases réduites de problèmes coercifs ou non coercifs est la définition de bornes fiables pour la constante de coercivité ou la constante inf-sup. Dans cette Note, nous généralisons et améliorons la méthode d'optimisation linéaire de contraintes successives présentées dans Huynh (2007), en proposant une version monotone de cet algorithme qui conduit à la fois à des évaluations plus stables et un nombre réduit de calculs *hors ligne*. ***Pour citer cet article : Y. Chen et al., C. R. Acad. Sci. Paris, Ser. I 346 (2008).***
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La méthode des bases réduites pour l'approximation de la solution $u(\nu)$ d'une famille de problèmes de la forme (1) dépendant de paramètres $\nu \in \mathcal{D}$ consiste tout d'abord à évaluer, par exemple par une méthode d'éléments finis sur un espace X^h de dimension $\mathcal{N}^h$, des solutions $u^h(\nu_i)$ pour un choix de paramètres ν_i appropriés et ensuite proposer une approximation de Galerkin dans l'espace $W^N = \text{Vect}\{u^h(\nu_1), \dots, u^h(\nu_N)\}$. Le problème ainsi formé est de complexité liée à $N \ll \mathcal{N}^h$, et si, par exemple, une hypotèse de décomposition affine sur a est faite du type

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$a(w, v; \nu) \equiv \sum_{q=1}^{Q} \Theta^q(\nu) a_q(w, v)$ et de même sur f, la méthode de bases réduites permet des calculs en ligne extrèmement rapides.

Pour fiabiliser ces méthodes, l'analyse a posteriori est un ingrédient essentiel. De façon classique, les estimations a posteriori, font apparaître au numérateur, une estimation du résidu du problème (1) et, au dénominateur, soit la constante d'ellipticité ou, à défaut la constante de la condition inf-sup (2), dont il est donc essentiel d'avoir une borne inférieure pertinente, et ce d'autant plus que le problème à résoudre est peu stable. Le problème qui nous intéresse entre dans ce cadre puisqu'il s'agit de connaître les solutions dans une cavité électromagnétique modélisée par les équations (3) dans le cadre de la géométrie donnée dans la Fig. 1. Ce problème, posé dans Ω où interviennent des paramètres $\epsilon|_{\Omega_i} = \epsilon_i$, $\mu|_{\Omega_i} = \mu_i$ pour $i = 1, 2$, possède des résonnances, c'est à dire des combinaisons de ces paramètres où la condition inf-sup est nulle.

Une méthode pour évaluer une telle borne inférieure par une méthode d'optimisation linéaire de contraintes successives a été présentée dans [2]. Nous proposons ici une version monotone de cet algorithme qui conduit à la fois à des évaluations plus stables et un nombre réduit de calculs *hors ligne*.

On ramène le cas où une condition inf-sup est satisfaite au cas où le problème est elliptique en remarquant que le carré de la condition inf-sup s'écrit comme la constante d'ellipticité d'un problème associé (voir (10)). On se concentre donc sur le cas elliptique. L'hypothèse de décomposition affine sur a permet d'écrire la constante d'ellipticité α^h selon (4), puis posant $y_q(w) = \frac{a_q(w,w)}{\|w\|_X^2}$ on obtient facilement (5). L'évaluation apparaît ainsi comme la minimisation d'une fonctionelle $\mathcal{J}$ sur un ensemble $\mathcal{Y}$ donné par (6). La méthode de [2] repose sur la définition de deux espaces $\mathcal{Y}_{UB} \subset \mathcal{Y} \subset \mathcal{Y}_{LB}$ de sorte que l'on obtient une borne inférieure et une borne supérieure en minimisant $\mathcal{J}$ sur ces ensembles comme indiqué en (8).

Notre définition de $\mathcal{Y}_{LB}$ proposée en (9) diffère de celle de [2] rappelée dans la remarque 1. Elle présente l'avantage important de rendre monotone l'algorithme glouton suivant :

Construction récursive de C_K : Pour des valeurs $M_\alpha \in \mathbb{N}$, $M_+ \in \mathbb{N}$, Ξ donnés et une tolérence $\epsilon_\alpha \in [0, 1]$ choisie, l'algorithme se déroule comme suit :

(1) Poser $K = 1$ et choisir $C_1 = \{w_1\}$ arbitrairement ;
(2) Trouver $w_{K+1} = \operatorname{argmax}_{\nu \in \Xi} \frac{\alpha_{UB}(\nu; C_K) - \alpha_{LB}(\nu; C_K)}{\alpha_{UB}(\nu; C_K)}$;
(3) Mettre à jour $C_{K+1} = C_K \cup w_{K+1}$;
(4) Répéter (2) et (3) jusqu'à ce que $\max_{\nu \in \Xi} \frac{\alpha_{UB}(\nu; C_{K_{\max}}) - \alpha_{LB}(\nu; C_{K_{\max}})}{\alpha_{UB}(\nu; C_{K_{\max}})} \leqslant \epsilon_\alpha$.

Les résultats numériques de la Fig. 2 illustrent bien le comportement monotone de $\frac{\alpha_{UB}(\nu, C_K) - \alpha_{LB}(\nu, C_K)}{\alpha_{UB}(\nu, C_K)}$ comme établis dans la Proposition 3.1. Il montrent aussi (au moins sur ce cas) que le nombre d'éléments dans C_K est notablement plus faible.

1. Introduction

In the context of optimization, design or optimal control, the numerical simulation of parametric problems of the form: find $u(\nu)$ in a Hilbert space X such that

$$a\big(u(\nu), v; \nu\big) = f(v; \nu), \forall v \in X, \tag{1}$$

for ν in a given parameter set $\mathcal{D}$ has to be done many times. The natural hypotheses over $a(w, v; \nu)$ that make problem (1) well-posed are uniform continuity and uniform inf-sup conditions. A finite element (FE) discretization of problem (1) is a standard way to approximate its solution $u^h(\nu) \simeq u(\nu)$ but it is most of the times infeasible to directly solve the finite element problem too many times because of the high marginal cost resulting from the large dimension of the discrete systems to be solved (equal to the dimension $\mathcal{N}^h$ of the finite element space X^h).

The reduced basis method (RBM) [5,1,4] has emerged as a very efficient and accurate method in this scenario. The fundamental observation that is recognized and exploited by RBM is the fact that the set of all solutions $\mathcal{M}^{\mathcal{D}} = \{u^h(\nu),\ \nu \in \mathcal{D}\}$ often has a small Kolmogorov width so that it can be very well approximated by a small-dimensional space, $W^N = \operatorname{span}\{u^h(\nu_1), \ldots, u^h(\nu_N)\}$, spanned by a few ($N \ll \mathcal{N}^h$) well suited solutions. The FE approximations corresponding to the parameters $\{\nu_1, \ldots, \nu_N\}$ can be done due to the small number N it should be performed.

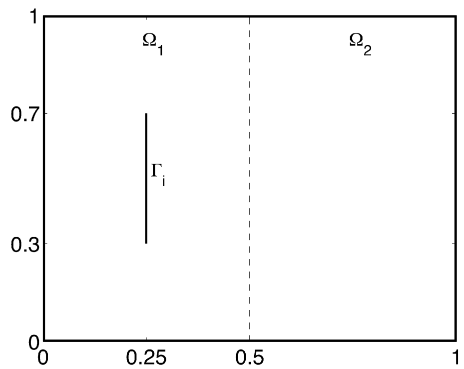

Fig. 1. The electromagnetic cavity problem.

For a given ν, the reduced basis method is then a Galerkin projection onto the N-dimensional approximation space W^N. At least when an affine decomposition holds on the (bi)linear forms e.g.: $a(w,v;\nu) \equiv \sum_{q=1}^{Q} \Theta^q(\nu) a_q(w,v)$, the complexity involves N only and not $\mathcal{N}^h$. In order to assure the fidelity of the RB solution $u^N(\nu)$ to approximate the FE solution $u^h(\nu)$, an a posteriori analysis is available but it relies deeply on an accurate knowledge of the discrete inf-sup condition of the bilinear form [3]:

$$\beta^h(\nu) \equiv \inf_{\omega \in X^h} \sup_{v \in X^h} \frac{a(\omega, v; \nu)}{\|\omega\|_X \|v\|_X}. \tag{2}$$

It is therefore crucial to find bounds for $\beta^h(\nu)$. Recently, D.B.P. Huynh et al. [2] proposed a constructive strategy for this purpose based on successive constraint method (SCM).

Our motivation comes from the problem that we want to solve: an electromagnetic cavity problem in Ω,

$$\begin{cases} -\epsilon\omega^2 E_x + \dfrac{1}{\mu}\dfrac{\partial}{\partial y}\left(\dfrac{\partial E_y}{\partial x} - \dfrac{\partial E_x}{\partial y}\right) = 0, & \text{in } \Omega, \\ -\epsilon\omega^2 E_y - \dfrac{1}{\mu}\dfrac{\partial}{\partial x}\left(\dfrac{\partial E_y}{\partial x} - \dfrac{\partial E_x}{\partial y}\right) = i\omega\cos\left(\omega\left(y - \dfrac{1}{2}\right)\right)\delta_{\Gamma_i}, & \text{in } \Omega, \end{cases} \tag{3}$$

with boundary condition $E_x\hat{n}_y - E_y\hat{n}_x = 0$ on $\partial\Omega$, $(\hat{n}_x, \hat{n}_y)$ being the unit outward normal of $\partial\Omega$ and Γ_i is as defined in Fig. 1. The parameters are $\epsilon|_{\Omega_i} = \epsilon_i$, $\mu|_{\Omega_i} = \mu_i$ for $i = 1, 2$ with the range of ϵ_i, μ_i to be specified later. The natural function space X is $H(\mathrm{curl})$.

In this Note, we study, improve and test the SCM on problem (3). Indeed, this problem gives rise to pure resonances, in which, mathematically, the inf-sup number goes to zero along the resonance lines, leading to the quest of fast and reliable evaluation of $\beta^h(\nu)$.

2. Successive constraints method—the coercive case

Bearing on the affine assumption made on the problem, the coercivity constant takes the expression

$$\alpha^h(\nu) \equiv \inf_{w \in X^h} \frac{a(w, w; \nu)}{\|w\|_X^2} = \inf_{w \in X} \sum_{q=1}^{Q} \Theta^q(\nu) \frac{a_q(w, w)}{\|w\|_X^2}. \tag{4}$$

Let us set $y_q(w) = \frac{a_q(w,w)}{\|w\|_X^2}$. First we recognize that

$$\alpha^h(\nu) = \inf_{w \in X^h} \sum_{q=1}^{Q} \Theta^q(\nu) y_q(w). \tag{5}$$

Second, by definition, $(y_1(w), \ldots, y_Q(w))$ belongs to the set

$$\mathcal{Y} \equiv \left\{ y = (y_1, \ldots, y_Q) \in \mathbb{R}^Q \mid \exists w \in X^h \text{ s.t. } y_q = y_q(w),\ 1 \leqslant q \leqslant Q \right\}, \tag{6}$$

so that the coercivity constant $\alpha^h(\nu)$ can be found by solving the following minimization problem

$$\alpha^h(\nu)=\min_{y\in\mathcal{Y}}\mathcal{J}(\nu;y),\quad \text{where } \mathcal{J}(\nu;y)=\sum_{q=1}^{Q}\Theta^q(\nu)y_q. \tag{7}$$

The idea of SCM is to build two sets $\mathcal{Y}_{LB}$ and $\mathcal{Y}_{UB}$ that are more amenable to the determination of

$$\alpha_{LB}(\nu;C_K)=\min_{y\in\mathcal{Y}_{LB}(\nu;C_K)}\mathcal{J}(\nu;y)\quad\text{and}\quad \alpha_{UB}(\nu;C_K)=\min_{y\in\mathcal{Y}_{UB}(C_K)}\mathcal{J}(\nu;y). \tag{8}$$

Assuming that in addition they are built so that $\mathcal{Y}_{UB}\subset\mathcal{Y}\subset\mathcal{Y}_{LB}$, we get a lower bound and an upper bound for $\alpha^h(\nu)$.

Proposition 2.1. *For given C_K, $(M_\alpha,M_+)\in\mathbb{N}^2$, and Ξ, $\alpha_{LB}(\nu;C_K)\leqslant\alpha^h(\nu)\leqslant\alpha_{UB}(\nu;C_K)$, $\forall\nu\in\mathcal{D}$.*

We follow here closely the method proposed in [2] for the construction of the two sets $\mathcal{Y}_{LB}$ and $\mathcal{Y}_{UB}$ and indicate the ingredient that makes the new approach monotonic. Let us set σ_q^-, σ_q^+ to be the minimum and maximum eigenvalues of a_q, that is, $\sigma_q^-\equiv\inf_{w\in X^h}y_q(w)$, $\sigma_q^+\equiv\sup_{w\in X^h}y_q(w)$, $1\leqslant q\leqslant Q$, and let $\mathcal{B}_Q\equiv\prod_{q=1}^{Q}[\sigma_q^-,\sigma_q^+]\subset\mathbb{R}^Q$. Obviously, $\mathcal{Y}\subset\mathcal{B}_Q$.

The definition of $\mathcal{Y}_{LB}$ and $\mathcal{Y}_{UB}$, involves two parameter sets $\Xi\equiv\{\nu_1\in\mathcal{D},\dots,\nu_J\in\mathcal{D}\}$ and $C_K\equiv\{w_1\in\mathcal{D},\dots,w_K\in\mathcal{D}\}$. Ξ could be the set of grid points of a mesh for the parameter domain and C_K is any subset of Ξ. Let $P_M(\nu;E)$ denote the M points closest to ν in E with E being Ξ or C_K.

We are now ready to define $\mathcal{Y}_{LB}$ and $\mathcal{Y}_{UB}$: for given C_K (and $M_\alpha\in\mathbb{N}$, $M_+\in\mathbb{N}$, and Ξ), we define

$$\begin{aligned}\mathcal{Y}_{LB}(\nu;C_K)\equiv\Bigg\{y\in\mathcal{B}_Q\ \Bigg|\ &\sum_{q=1}^{Q}\Theta^q(\nu')y_q\geqslant\alpha^h(\nu'),\ \forall\nu'\in P_{M_\alpha}(\nu;C_K);\\ &\sum_{q=1}^{Q}\Theta^q(\nu')y_q\geqslant\alpha_{LB}(\nu',C_{K-1}),\ \forall\nu'\in P_{M_+}(\nu;\Xi\backslash C_K)\Bigg\},\end{aligned} \tag{9}$$

with the natural definition $\alpha_{LB}(\nu,C_0)\equiv 0,\forall\nu\in\Xi$. Next we set, as in [2], $\mathcal{Y}_{UB}(C_K)\equiv\{y^*(w_k),1\leqslant k\leqslant K\}$ for $y^*(\nu)\equiv\operatorname{argmin}_{y\in\mathcal{Y}}\mathcal{J}(\nu;y)$. Obviously, $\mathcal{Y}_{UB}\subset\mathcal{Y}\subset\mathcal{Y}_{LB}$, so Proposition 2.1 holds.

Note that (8) is in fact a Linear Program (LP) that contains Q design variables and $2Q+M_\alpha+M_+$ one-sided inequality constraints: the operation count for the on-line stage $\nu\to\alpha_{LB}(\nu)$ is independent of $\mathcal{N}^h$.

3. Greedy definition of the constraints set C_K

It only remains to construct C_K, this is done by an off-line "greedy" algorithm. Given $M_\alpha\in\mathbb{N}$, $M_+\in\mathbb{N}$, Ξ, and a tolerance $\epsilon_\alpha\in[0,1]$, the algorithm reads:

(1) Set $K=1$ and choose $C_1=\{w_1\}$ arbitrarily;
(2) Find $w_{K+1}=\operatorname{argmax}_{\nu\in\Xi}\frac{\alpha_{UB}(\nu;C_K)-\alpha_{LB}(\nu;C_K)}{\alpha_{UB}(\nu;C_K)}$;
(3) Update $C_{K+1}=C_K\cup w_{K+1}$;
(4) Repeat (2) and (3) until $\max_{\nu\in\Xi}\frac{\alpha_{UB}(\nu;C_{K_{\max}})-\alpha_{LB}(\nu;C_{K_{\max}})}{\alpha_{UB}(\nu;C_{K_{\max}})}\leqslant\epsilon_\alpha$.

We note that the off-line computations are

(1) $2Q+K_{\max}$ eigenproblems to form $\mathcal{B}_Q$ and to obtain $y^*(w_k)$, $\alpha^h(w_K)$;
(2) $O(\mathcal{N}QK_{\max})$ operations to form $\mathcal{Y}_{UB}$;
(3) $JK_{\max}$ LPs of size $O(Q+M_\alpha+M_+)$.

This algorithm is monotonic as is stated in the following

Proposition 3.1. *With $\mathcal{Y}_{LB}(\nu; C_K)$ and the greedy algorithm defined as above, we have for any $\nu \in \Xi$, as K increases,*

(1) $\alpha_{LB}(\nu, C_K)$ *is non-decreasing*;
(2) $\alpha_{UB}(\nu, C_K)$ *decreases*;
(3) $\frac{\alpha_{UB}(\nu,C_K)-\alpha_{LB}(\nu,C_K)}{\alpha_{UB}(\nu,C_K)}$ *decreases.*

Proof. (1) simply follows from the fact that for any $\nu \in \Xi$, $\sum_{q=1}^{Q} \Theta^q(\nu) y_q \geqslant \alpha_{LB}(\nu, C_{K-1})$ is included as one constraint when we are looking for $\alpha_{LB}(\nu, C_K)$. This means the *updated* lower bound is getting no smaller. (2) is a direct consequence of the definition of $\alpha_{UB}(\nu, C_K)$, and (3) follows from (1) and (2). □

Remark 1. The simple but radical difference with respect to the original definition in [2] is in the definition of $\mathcal{Y}_{LB}(\nu; C_K)$ that was originally

$$\mathcal{Y}_{LB}^{\text{old}}(\nu; C_K) \equiv \left\{ y \in \mathcal{B}_Q \;\Big|\; \sum_{q=1}^{Q} \Theta^q(\nu') y_q \geqslant \alpha^h(\nu'),\ \forall \nu' \in P_{M_\alpha}(\nu; C_K);\ \sum_{q=1}^{Q} \Theta^q(\nu') y_q \geqslant 0,\ \forall \nu' \in P_{M_+}(\nu; \Xi) \right\}.$$

Remark 2. The *new* SCM is more efficient than the original formulation, both off-line and on-line, in the following two ways:

(1) The new method is likely to solve less eigenproblems than the original SCM to get a lower bound of the same quality. This is because, on the same set C_K, the new method provides larger lower bound. The final set C_K that meets the stopping criteria is going to be no larger than that for the original method.
(2) With the same settings, the new method shall take less time to solve since the constraints are stricter.

4. Successive constraints method—the non-coercive case

For the non-coercive case, we introduce the operators $T^q,\ T^\nu : X^h \to X^h$ as follows

$$\left(T^q w, v\right)_X = a_q(w, v), \quad \forall v \in X^h,\ 1 \leqslant q \leqslant Q \quad \text{and} \quad T^\nu w \equiv \sum_{q=1}^{Q} \Theta^q(\nu) T^q w.$$

It is then easy to show that $(\beta^h(\nu))^2 = \inf_{w \in X^h} \frac{(T^\nu w, T^\nu w)_X}{\|w\|_X^2}$ which can be expanded as

$$\left(\beta^h(\nu)\right)^2 = \inf_{w \in X^h} \sum_{q'=1}^{Q} \sum_{q''=q'}^{Q} (2 - \delta_{q'q''}) \Theta^{q'}(\nu) \Theta^{q''}(\nu) \frac{(T^{q'} w, T^{q''} w)_X}{\|w\|_X^2},$$

where, $\delta_{q'q''}$ is the Kronecker delta. Based on the following formal identifications

$$(2 - \delta_{q'q''}) \Theta^{q'}(\nu) \Theta^{q''}(\nu), \quad 1 \leqslant q' \leqslant q'' \leqslant Q \equiv \hat{\Theta}^q(\nu),\ 1 \leqslant q \leqslant \hat{Q} \equiv \frac{Q(Q+1)}{2},$$

$$\left(T^{q'} w, T^{q''} w\right)_X, \quad 1 \leqslant q' \leqslant q'' \leqslant Q \equiv \hat{a}_q(w, v),\ 1 \leqslant q \leqslant \hat{Q},$$

we observe that

$$\left(\beta^h(\nu)\right)^2 \equiv \inf_{w \in X^h} \sum_{q=1}^{\hat{Q}} \hat{\Theta}^q(\nu) \frac{\hat{a}_q(w, w)}{\|w\|_X^2}, \tag{10}$$

so that $(\beta^h(\nu))^2$ can be interpreted as an equivalent coercivity constant$(\beta^h(\nu))^2 \equiv \hat{\alpha}^h(\nu)$. We may then directly apply our SCM procedure to (10).

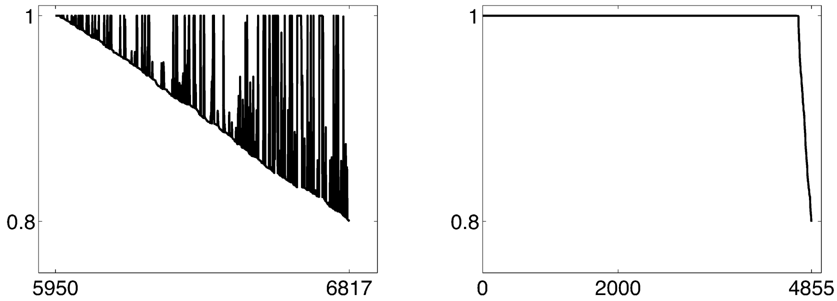

Fig. 2. Two-dimensional case: A plot of $\frac{\alpha_{UB}-\alpha_{LB}}{\alpha_{UB}}$ in the greedy algorithm. (Left) with the original SCM procedure, (right) with the monotonic approach described in this paper.

5. Numerical results

Here, we show numerical results for two parameters, $\epsilon_2 \in [2, 6]$, $\mu_2 \in [1.0, 1.2]$. We set $\epsilon_1 = 1$, $\mu_1 = 1$, $\omega = \frac{5\pi}{2}$, $M_\alpha = 20$, $M_+ = 6$, $C_1 = \{(2.0, 1.0)\}$, $\epsilon_\alpha = 0.8$ and Ξ is a uniform Cartesian grid of 513×33. This is a pure resonance case in which the inf-sup number goes to zero along the resonance lines.

We first run the original SCM and obtain 6817 points in the parameter space to compute the bounds. We plot in Fig. 2(left) the quantity $\max_{\nu\in\Xi} \frac{\alpha_{UB}(\nu,C_K)-\alpha_{LB}(\nu,C_K)}{\alpha_{UB}(\nu,C_K)}$ for $K = 5950, \ldots, 6817$ (it equals one for $K < 5950$) and note it is oscillating although it gets below ϵ_α eventually.

As a comparison, we run the new SCM with exactly the same settings and obtain only 4855 points. In addition to the fact that the final set of selected points is much smaller, the quantity $\max_{\nu\in\Xi} \frac{\alpha_{UB}(\nu,C_K)-\alpha_{LB}(\nu,C_K)}{\alpha_{UB}(\nu,C_K)}$ decreases until smaller than ϵ_α monotonically, as shown in Fig. 2(right).

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