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Partial Differential Equations

Observation of some elastic networks

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Abstract

We consider a network of vibrating elastic strings. Using a generalized Poisson formula and some Tauberian theorem, we give a Weyl formula with optimal remainder estimate. As a consequence we prove some observability and stabilization results. ***To cite this article: K. Ammari, M. Dimassi, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Observation de certains réseaux élastiques. Nous considérons un réseau de cordes. En utilisant une formule de Poisson généralisée et un théorème Taubérien nous prouvons une formule de Weyl avec reste optimal. Comme conséquence nous prouvons un résultat d'observablité et de stabilisation. ***Pour citer cet article : K. Ammari, M. Dimassi, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Version française abrégée

Nous établissons une formule de Weyl avec reste optimal pour le laplacien sur un graphe (voir la section suivante pour la définition exacte de l'opérateur Δ_G). Notons par $N_{\Delta_G}(\lambda)$ le nombre des valeurs propres (comptés avec leurs multiplicité) dans l'intervalle $]-\infty, \lambda]$.

Notre résultat principal est :

Théorème. *Il existe $\lambda_0 \gg 1$ tel que $N_{\Delta_G}(\lambda) = \frac{L}{\pi}\sqrt{\lambda} + \mathcal{O}(1)$, uniformément par rapport à $\lambda \in]\lambda_0, +\infty[$.*

Le résultat ci-dessus généralise en particulier l'asymptotique de Weyl obtenue par Nicaise dans [6].

1. Introduction

Let Γ be a connected topological graph embedded in $\mathbb{R}^m$, $m \in \mathbb{N}^*$, with n vertices $\mathcal{S} = \{E_i, 1 \leqslant i \leqslant n\}$ and N edges $\mathcal{A} = \{k_i, 1 \leqslant i \leqslant N\}$. Each edge k_j is a Jordan curve in $\mathbb{R}^m$ and is assumed to be parametrized by its arc length

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parameter x_j, such that the parametrization

$$\pi_j : [0, l_j] \to k_j : x_j \mapsto \pi_j(x_j)$$

is $C^\nu([0, l_j], \mathbb{R}^m)$ for all $1 \leqslant j \leqslant N$.

We now define the C^ν-network G associated with Γ as the union

$$G = \bigcup_{j=1}^{N} k_j.$$

The incidence matrix $D = (d_{ij})_{n\times N}$ of Γ is defined by

$$d_{ij} = \begin{cases} 1 & \text{if } \pi_j(l_i) = E_i, \\ -1 & \text{if } \pi_j(0) = E_i, \\ 0 & \text{otherwise.} \end{cases}$$

The adjacency matrix $\mathcal{E} = (e_{ih})_{n\times n}$ of Γ is given by

$$e_{ih} = \begin{cases} 1 & \text{if there exists an edge } k_{s(i,h)} \text{ between } E_i \text{ and } E_h, \\ 0 & \text{otherwise.} \end{cases}$$

The valence of the node E_i will be noted $\gamma(E_i)$. There are two types of nodes: the interior nodes $\operatorname{int}\mathcal{S} = \{E_i \in \mathcal{S};\ \gamma(E_i) > 1\}$ and the boundary nodes $\partial\mathcal{S} = \{E_i \in \mathcal{S}\colon\ \gamma(E_i) = 1\}$. In the following we will denote $I_{\text{ext}} = \{i \in \{1,\dots,n\}\colon\ \gamma(E_i) = 1\}$ and $I_{\text{int}} = \{1,\dots,n\} \setminus I_{\text{ext}}$. We denote by $N_i = \{j \in \{1,\dots,n\}, E_i \in k_j\}$ the set of edges adjacent to E_i. We remark that if $E_i \in \partial\mathcal{S}$, then N_i is a singleton which denotes by $\{j_i\}$.

For a function $u : G \to \mathbb{R}$, we set $u_j = u \circ \pi_j : [0, l_j] \to \mathbb{R}$, its restriction to the edge k_j. We further use the abbreviations:

$$u_j(E_i) = u_j\big(\pi_j^{-1}(E_i)\big), \qquad u_{jx_j^{(n)}}(E_i) = \frac{\mathrm{d}^n u_j}{\mathrm{d}x_j^n}\big(\pi_j^{-1}(E_i)\big), \quad n \in \mathbb{N}^*.$$

Finally, differentiations are carried out on each edge k_j with respect to the arc length parameter x_j.

We consider the following operator Δ_G on the Hilbert space $H = \prod_{j=1}^N L^2(0, l_j)$, endowed with the usual product norm.

$$D(\Delta_G) = \big\{u \in H, u_j \in H^2(0, l_j) \text{ satisfying (1)–(3)}\big\}, \qquad \Delta_G u = (-u_{jx_j^{(2)}})_{j=1}^N, \quad \forall u \in D(\Delta_G).$$

If $O = (O_{ih})_{n\times n}$ is the orientation matrix defined by

$$O_{ih} = \begin{cases} 1 & \text{if } k_{s(i,h)} \text{ is directed from } E_i \text{ to } E_h, \\ -1 & \text{if } k_{s(i,h)} \text{ is directed from } E_h \text{ to } E_i, \\ 0 & \text{else,} \end{cases}$$

$$u \text{ is continuous on } G, \tag{1}$$

$$\sum_{j=s(i,h)\in N_i} O_{ih} u_{jx_j}(E_i) = 0, \quad \forall i = 1,\dots,n, \tag{2}$$

$$u_{j_i}(E_i) = 0, \quad \forall i \in I_{\text{ext}}. \tag{3}$$

We study a model of networks of strings. More precisely we consider the following initial problems: on a finite network, of length L, made of edges k_j, identified to a real interval of length l_j, $j = 1,\dots,N$ (i.e. $L = \sum_{i=1}^N l_i$) we consider the eigenvalue problem

$$-\frac{\mathrm{d}^2 u_j}{\mathrm{d}x_j^2} = \lambda u_j, \quad k_j, j = 1,\dots,N, \ u \text{ satisfies (1)–(3).} \tag{4}$$

In the present Note we give some asymptotic Weyl formula of some networks.

The plan of the paper is as follows. In the following section we give precise statements of the main results. The last section is devoted to some application.

2. Asymptotic with optimal remainder estimate

Let $\lambda_0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n \leqslant \cdots$ be the eigenvalues, repeated according to their multiplicity, of the self-adjoint operator Δ_G on a C^2-network G which is defined below.

We introduce the counting function of eigenvalues: $N_{\Delta_G}(\lambda) := \#\sigma(\Delta_G) \cap]-\infty, \lambda]$, where in general $\#A$ denotes the number of elements of A.

Our main result can now be stated as follows. The proof is based on a Poisson formula (see [1] and [7]) and some standard Tauberian argument.

Theorem 1. *There exists $\lambda_0 \gg 1$ such that $N_{\Delta_G}(\lambda) = \frac{L}{\pi}\sqrt{\lambda} + \mathcal{O}(1)$, uniformly on $\lambda \in]\lambda_0, +\infty[$.*

3. Application to a polynomial stabilization of a star-shaped network of strings

We consider the following initial and boundary value problems:

$$\frac{\partial^2 u_i}{\partial t^2}(x,t) - \frac{\partial^2 u_i}{\partial x^2}(x,t) = 0, \quad u_i(x,0) = u_i^0(x), \quad \frac{\partial u_i}{\partial t}(x,0) = u_i^1(x), \quad 0 < x < l_i,\ t > 0, \tag{5}$$

$$u_i(l_i,t) = 0, \quad u_i(0,t) = u_j(0,t), \quad \sum_{i=1}^N \frac{\partial u_i}{\partial x}(0,t) = \frac{\partial u_1}{\partial t}(0,t), \quad t > 0, \tag{6}$$

for $i, j = 1, \ldots, N$ and where $u_i : [0, l_i] \times (0, +\infty) \to \mathbb{R},\ i = 1, \ldots, N,\ N \geqslant 2$ be the displacement of the string of length l_i. Denote by $L = \sum_{i=1}^N l_i$. Let $H = \prod_{i=1}^N L^2(0, l_i),\ A_1 = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}$,

$$\mathcal{D}(A_1) = \left\{ (u_1, \ldots, u_N) \in \prod_{i=1}^N H^2(0, l_i), u_i(0) = u_j(0),\ u_i(l_i) = 0,\ \sum_{i=1}^N \frac{\mathrm{d}u_i}{\mathrm{d}x}(0) = 0 \right\}.$$

Let $B_1 \in \mathcal{L}(\mathbb{R}, \mathcal{D}(A_1^{1/2})')$, $B_1 v = (A_1)_{-1}\mathcal{N} v, \forall v \in \mathbb{R}$, where $(A_1)_{-1} : \mathcal{D}(A_1^{1/2}) \to \mathcal{D}(A_1^{1/2})'$ is an extension of A_1 to $\mathcal{D}(A_1^{1/2})$, $\mathcal{N} \in \mathcal{L}(\mathbb{R}, \mathcal{D}(A_1^{1/2}))$ and $\mathcal{N} v$ is a solution of:

$$\frac{\mathrm{d}^2(\mathcal{N}v)_i}{\mathrm{d}x^2} = 0, \quad (\mathcal{N}v)_i(l_i) = 0, \quad (\mathcal{N}v)_i(0) = (\mathcal{N}v)_j(0), \quad \sum_{i=1}^N \frac{d(\mathcal{N}v)_i}{dx}(0) = v, \quad 0 < x < l_i, \tag{7}$$

for all $i, j = 1, \ldots, N$, and $B_1^* \psi = \psi_1(0)$, for all $\psi \in \mathcal{D}(A_1^{1/2})$. We denote by $\lambda_k = \mu_k^2$ the eigenvalues of A_1. In the case: $l_i / l_j \notin \mathbb{Q}$, for all i, j, $1 \leqslant i \neq j \leqslant N$, the eigenvalues λ_k are simple (see [2]) and the corresponding eigenfunctions are given by:

$$\left(\phi_k^1, \ldots, \phi_k^N\right), \quad \phi_k^i(x) = \frac{\sin(\mu_k(x - l_i))}{\sin(\mu_k l_i)(\sum_{i=1}^N l_i / (\sin^2(\mu_k l_i)))^{1/2}}, \quad i = 1, \ldots, N.$$

We define the energy of a $(u_1, \ldots, u_N)$, solution of (5), (6), at instant t by

$$E(t) = \sum_{i=1}^N \frac{1}{2} \int_0^{l_i} \left(\left| \frac{\partial u_i}{\partial t}(x,t) \right|^2 + \left| \frac{\partial u_i}{\partial x}(x,t) \right|^2 \right) \mathrm{d}x. \tag{8}$$

The well-posedness space for (5), (6) is $E = V \times \prod_{i=1}^N L^2(0, l_i)$, where

$$V = \left\{ \phi = (\phi_i)_{i=1,\ldots,N}, \phi_i \in H^1(0, l_i) \mid \phi(l_i) = 0,\ \phi_i(0) = \phi_j(0) \right\}.$$

Denote

$$\mathcal{D}(\mathcal{A}_d) = \left\{ \left((u_i)_{i=1,\ldots,N}, (v_i)_{i=1,\ldots,N}\right) \in \left[V \cap \prod_{i=1}^N H^2(0, l_i) \right] \times V, \sum_{i=1}^N \frac{\mathrm{d}u_i}{\mathrm{d}x}(0) = v_1(0) \right\}. \tag{9}$$

The corresponding operator $\mathcal{A}_d$ is defined by the same expression as A_1.

If $(u^0, u^1) \in E$, then the problem (5), (6) admits a unique solution $(u, \frac{\partial u}{\partial t}) \in C(0, +\infty; V \times \prod_{i=1}^N L^2(0, l_i))$ and we have: $\lim_{t \to +\infty} E(t) = 0$ holds true for any finite energy solution of (5), (6) if and only if $l_i/l_j \notin \mathbb{Q}, \forall 1 \leqslant i \neq j \leqslant N$, where $\mathbb{Q}$ is the set of all rational numbers. Denote by $\mathcal{S}$ the set of all numbers ρ such that $\rho \notin \mathbb{Q}$ and if $[0, a_1, \ldots, a_n, \ldots]$ is the expansion of ρ as a continued fraction, then (a_n) is bounded. Let us notice that $\mathcal{S}$ is obviously uncountable and, by classical results on Diophantine approximation, its Lebesgue measure is equal to zero. Roughly speaking the set $\mathcal{S}$ contains the irrationals which are approximable by rational numbers. In particular, by Euler–Lagrange theorem $\mathcal{S}$ contains all l_i/l_j, $1 \leqslant i \neq j \leqslant N$, such that l_i/l_j is an irrational quadratic number (i.e. satisfying a second degree equation with rational coefficients). According to [5], we have that $l_i/l_j \in \mathcal{S}$, $1 \leqslant i \neq j \leqslant N$, if and only if there exists a positive constant C such that: $|||\frac{l_i}{l_j} m||| := \min_{\frac{l_i}{l_j} m - x \in \mathbb{Z}} |x| \geqslant \frac{C}{m}$, $\forall m \in \mathbb{N}^*$.

Corollary 2.

1. *If $l_i/l_j \in \mathcal{S}$, $\forall 1 \leqslant i \neq j \leqslant N$, there exists $\beta > 0$ such that for all $t \geqslant 0$ we have*
$$E(t) \leqslant \frac{C}{(t+1)^{1/\beta}} \left\| \left(u^0, u^1\right) \right\|^2_{\mathcal{D}(\mathcal{A}_d)}, \quad \forall \left(u^0, u^1\right) \in \mathcal{D}(\mathcal{A}_d), \tag{10}$$
where $C > 0$ is a constant depending only on l_i, $i = 1, \ldots, N$.
2. *For all $\varepsilon > 0$ there exists a set $B_\varepsilon \subset \mathbb{R}$, such that the Lebesgue measure of $\mathbb{R} \setminus B_\varepsilon$ is equal to zero, and constants β, $C_\varepsilon > 0$ for which, if $l_i/l_j \in B_\varepsilon$, $1 \leqslant i \neq j \leqslant N$, then for all $t \geqslant 0$*
$$E(t) \leqslant \frac{C_\epsilon}{(t+1)^{1/(\beta+\varepsilon)}} \left\| \left(u^0, u^1\right) \right\|^2_{\mathcal{D}(\mathcal{A}_d)}, \quad \forall \left(u^0, u^1\right) \in \mathcal{D}(\mathcal{A}_d), \tag{11}$$
where $C_\varepsilon > 0$ is a constant depending only on l_i, $i = 1, \ldots, N$, and ε.

The proof of Corollary 2 needs the following lemma:

Lemma 3. *(See [2, Ammari–Jellouli].) Let $\gamma > 0$ be a fixed real number and $C_\gamma = \{\lambda \in \mathbb{C} \mid \mathrm{Re}(\lambda) = \gamma\}$. Then, the function $f(\lambda) = 1/\sum_{i=1}^N \coth(\lambda l_i)$, is bounded on C_γ.*

Proof of Corollary 2. For $k < 0$, we denote by $\mu_k = -\mu_{-k}$. Let $0 < \eta' \leqslant \eta$ with $\eta' \leqslant 2\mu_1/M$. We claim that
$$\mu_{k+M} - \mu_k \geqslant \eta' M, \quad \forall k \in \mathbb{Z}. \tag{12}$$
In fact, for $k > 0$ resp. $(k + M < 0)$ (12) follows from Theorem 1 resp. (Theorem 1 and the fact that $\mu_k = -\mu_{-k}$). For $k + M > 0$ and $k < 0$ we use that $\mu_{k+M} - \mu_k = \mu_{k+M} + \mu_{-k} \geqslant 2\mu_1 \geqslant M\eta'$.

We denote by A_j, $j = 1, \ldots, M$, the set of integers m satisfying:
$$\mu_m - \mu_{m-1} \geqslant \eta',$$
$$\mu_n - \mu_{n-1} < \eta', \quad \forall m+1 \leqslant n \leqslant m+j-1,$$
$$\mu_{m+j} - \mu_{m+j-1} \geqslant \eta'.$$
Then the $M(M+1)/2$ sets $A_j + k = \{n + k;\ n \in A_j\}$, $0 \leqslant k < j \leqslant M$, are disjoint and form a partition of the set $\mathbb{Z}$. Let us introduce for $m \in A_j$ the divided differences $e_m(t), \ldots, e_{m+j-1}(t)$ of the exponential functions $\mathrm{e}^{\mathrm{i}\mu_n t}$, $n = m, \ldots, m+j-1$. Since μ_k are simple (see [2]) then e_k, $k = m, \ldots, m+j-1$ (see [4]) is given by the following expression $e_k(t) = \sum_{p=m}^k [\prod_{q=m,\ q \neq p}^k (\mu_p - \mu_q)]^{-1} \mathrm{e}^{\mathrm{i}\mu_p t}$. For $(u_1^0, \ldots, u_N^0, u_1^1, \ldots, u_N^1)^t = \sum_{k \geqslant 0} (a_k (\frac{1}{\mu_k} \phi_k^1, \ldots, \frac{1}{\mu_k} \phi_k^N, \phi_k^1, \ldots, \phi_k^N)^t + a_{-k} (-\frac{1}{\mu_k} \phi_k^1, \ldots, -\frac{1}{\mu_k} \phi_k^N, \phi_k^1, \ldots, \phi_k^N)^t)$, $(a_k)_k \in l^2$, we have
$$\frac{\partial \varphi_1}{\partial t}(0, t) = \sum_{k \geqslant 0} \left(a_k \mathrm{e}^{\mathrm{i}\mu_k t} + a_{-k} \mathrm{e}^{-\mathrm{i}\mu_k t}\right) \phi_k^1(0) = \sum_{k \geqslant 0} \left(b_k e_k(t) + b_{-k} e_{-k}(t)\right),$$
where $\varphi = (\varphi_1, \ldots, \varphi_N)^t$ is a solution of conservative system associated to (5), (6).

According to [4, Theorem 9.4] we have that for $T > 2\pi/\eta$ there exists a constant $C_1 > 0$ such that $\int_0^T |\frac{\partial\varphi_1}{\partial t}(0,t)|^2\,\mathrm{d}t \geqslant C_1 \sum_{k\geqslant 0}(|b_k|^2 + |b_{-k}|^2)$. If $l_i/l_j \in \mathcal{S}$, $\forall 1 \leqslant i \neq j \leqslant N$, there exist $\beta, C_2 > 0$ such that we have: $\int_0^T |\frac{\partial\varphi_1}{\partial t}(0,t)|^2\,\mathrm{d}t \geqslant C_2 \sum_{k\geqslant 0} |\mu_k|^{-\beta}(|a_k|^2 + |a_{-k}|^2)$. Which implies, according to [3, Theorem 2.4], the estimate (10).

In order to prove (11) we use a well-known result asserting that for all $\varepsilon > 0$, there exists a set $B_\varepsilon \subset \mathbb{R}$, such that the Lebesgue measure of $\mathbb{R} \setminus B_\varepsilon$ is equal to zero. If $l_i/l_j \in B_\varepsilon$, $1 \leqslant i \neq j \leqslant N$, there exists a constant $C > 0$ such that $|||ml_i/l_j||| \geqslant C/m^{1+\varepsilon}$, $\forall m \geqslant 1$. Then, as above we have for $T > 2\pi/\eta$ that there exists a constant $C_3 > 0$ such that:

$$\int_0^T \left|\frac{\partial\varphi_1}{\partial t}(0,t)\right|^2 \mathrm{d}t \geqslant C_3 \sum_{k\geqslant 0} |\mu_k|^{-\beta-\varepsilon}\big(|a_k|^2 + |a_{-k}|^2\big).$$

Which implies (11), according to [3, Theorem 2.4]. □

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