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Partial Differential Equations

Global Cauchy problems for hyperbolic systems with characteristics admitting superlinear growth for $|x| \to \infty$

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Abstract

We investigate the global well-posedness of the Cauchy problem for first order linear hyperbolic systems allowing *superlinear* growth of the characteristic roots for $|x| \to +\infty$. We introduce hypotheses on the superlinear growth inspired by theorems of A. Wintner in 1945 for global solutions of ODEs and show global well-posedness of the Cauchy problem in two types of new weighted Sobolev spaces. We construct a change of the space variables of a global Liouville type which reduces to the case of bounded coefficients. As an outcome, we derive finite propagation speed and the existence of finite domains of dependence for hyperbolic systems of differential equations. We exhibit also instant blow-up of solutions near $t = 0$ and provide explicit examples proving that our estimates are sharp. ***To cite this article: D. Gourdin, T. Gramchev, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Problème de Cauchy global pour des systèmes hyperboliques á coefficients superlinéaire lorsque $|x| \to +\infty$. Nous étudions les problèmes de Cauchy bien posès pour les systèmes hyperboliques linéaires du 1er ordre avec des racines caractéristiques superlinéaires lorsque $|x| \to \infty$. On introduit des hypothèses sur la croissance superlinéaire inspirée des théorèmes de A. Wintner en 1945 pour les solutions globales d'équations différentielles ordinaires et nous montrons que le résultat est acquis pour deux types d'espaces de Sobolev avec poids. Nous construisons une transformation du type de Liouville globale des variables d'espace qui réduit le problème au cas de coefficients bornés. En consequence nous montrons la propagation a vitesse finie et l'existence d'un domaine fini de dépendance. On montre le blow up des solutions près de $t = 0$, avec des exemples explicites montrant que les estimations sont pointues. ***Pour citer cet article : D. Gourdin, T. Gramchev, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

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Version française abrégée

Le notations et definitions utilisées ici sont précisées dans la version anglaise. On considère le problème de Cauchy pour les systèmes hyperboliques du premierordre, á coefficients non bornés superlinéaires en x lorsque $|x| \to \infty$

$$\partial_t u = \mathrm{i}A(t,x,D_x)u + b(t,x,D_x)u + f(t,x), \quad u(0,x) = u^0(x), \quad t \in \mathbb{R},\ x \in \mathbb{R}^n, \tag{1}$$

où

$$A(t,x,D_x) = \{A_{j,k}(t,x,D_x)\}_{j,k=1}^m \quad \text{et} \quad b(t,x,D_x) = \{b_{j,k}(t,x,D_x)\}_{j,k=1}^m$$

sont des matrices d'opérateurs différentiels (resp. pseudo-différentiels) en x d'ordres respectifs 1 et 0, dépendant régulièrement de $t \in \mathbb{R}$. On suppose que :

(H1) $A(t,x,\xi)$ admet des racines caractéristiques $\lambda_1(t,x,\xi), \ldots, \lambda_m(t,x,\xi)$ réelles dans $C^\infty(\mathbb{R}^{n+1} \times \mathbb{R}^n \setminus 0)$, positivement homogènes de degré 1 relativement à $\xi \in \mathbb{R}^n \setminus 0$.

(H2) (croissance superlinéaire lorsque $|x| \to +\infty$ et condition de Wintner) Il existe pour chaque $T > 0$, une fonction positive régulière $\rho = \rho_T$ sur $[0, +\infty[$ avec

$$\int_1^{+\infty} \frac{1}{y\rho(y)}\,\mathrm{d}y = +\infty, \qquad \sup_{r \geqslant 0} \left| \frac{r^k D_r^k \rho(r)}{\rho(r)} \right| < +\infty, \quad \forall k \in \mathbb{Z}_+ \tag{2}$$

et pour chaque $\ell \in \mathbb{Z}_+$, α et β dans $\mathbb{Z}_+^n$, une constante $C = C_{T\alpha\beta} > 0$ tel que

$$\max_{j=1,\ldots,m} \left| D_t^\ell D_x^\alpha D_\xi^\beta \lambda_j(t,x,\xi) \right| \leqslant C\langle x\rangle^{1-|\alpha|} \rho(\langle x\rangle) |\xi|^{1-|\beta|}, \quad t \in [-T,T],\ x \in \mathbb{R}^n,\ \xi \in \mathbb{R}^n \setminus 0, \tag{3}$$

$$\left\| D_t^\ell D_x^\alpha D_\xi^\beta b(t,x,\xi) \right\| \leqslant C\langle x\rangle^{-|\alpha|} \rho(\langle x\rangle) \langle\xi\rangle^{-|\beta|}, \quad t \in [-T,T],\ x \in \mathbb{R}^n,\ \xi \in \mathbb{R}^n. \tag{4}$$

(H3) Il existe $S_0(t,x,\xi)$ matrice $m \times m$ d'éléments vérifiant (4) diagonalisant $A(t,x,\xi)$ sous forme de la réduction diagonale des racines caractéristiques λ_j $(1 \leqslant j \leqslant m)$.

Avec le cadre fonctionnel défini par (H4) on montre que le problème de Cauchy (1) est bien posé dans $CH_T^{s,\mathcal{W}}(\mathbb{R}^n)$ et la solution vérifie l'inegalité d'énergie (8) lorsque A et b sont différentiels. De plus, dans le cas pseudo-différentiel, si $u^0 \in \mathcal{S}(\mathbb{R}^n)$ et $f \in C(\mathbb{R} : \mathcal{S}(\mathbb{R}^n))$, alors $u \in C(\mathbb{R} : \mathcal{S}(\mathbb{R}^n))$ et (8) reste valable pourvu que, en plus des hypothèses, on ait

$$\sup_{y \in \mathbb{R}} \frac{\rho(|y|)}{\ln(2+|y|)} < +\infty. \tag{i}$$

La condition (i) est pointue pour les systèmes differentiels.

Sous la condition de Wintner, on montre que l'on peut ramener le cas de la croissance (super)linéaire au cas des coefficients bornés, grace à la construction d'un difféomorphisme de type de Liouville global φ de $\mathbb{R}^n$. Le problème de Cauchy est alors bien posé dans $L^2\varphi$ avec $\|u\|_{L^2_\varphi} = \|\varphi u\|_{L^2}$. L'existence d'un tel difféomorphisme global est un fait et un résultat nouveaux en soi utilisant une variante du theorème de Borel précisé. En application, nous montrons la vitesse finie de propagation et l'existence d'un domaine de dépendance.

1. Introduction

We study the Cauchy problem for the first order hyperbolic systems

$$\partial_t u = \mathrm{i}A(t,x,D_x)u + b(t,x,D_x)u + f(t,x), \quad u(0,x) = u^0(x), \quad t \in \mathbb{R},\ x \in \mathbb{R}^n, \tag{1}$$

where $A(t,x,D_x) = \{A_{jk}(t,x,\xi)\}_{j,k=1}^m$ (respectively, $b(t,x,D_x) = \{b_{jk}(t,x,D_x)\}_{j,k=1}^m$) is an $m \times m$ matrix valued first (respectively, zero) order differential or pseudodifferential operators in x depending smoothly on $t \in \mathbb{R}$. Here we use the traditional notation $D_x = \mathrm{i}^{-1}\partial_x$.

The fundamental hypotheses are he following ones:

(H1) (hyperbolicity with smooth characteristics) We assume that $A(t,x,\xi)$ admits real characteristics roots $\lambda_1(t,x,\xi), \ldots, \lambda_m(t,x,\xi)$ belonging to $C^\infty(\mathbb{R}^{n+1} \times \mathbb{R}^n \setminus 0)$ positively homogeneous of order 1 with respect to $\xi \in \mathbb{R}^n \setminus 0$.

(H2) (superlinear growth for $|x| \to +\infty$ under a Wintner type condition cf. [11], see also [7]). We require that for every $T > 0$ there exists a positive smooth function $\rho = \rho_T$ defined on $[0, +\infty[$ satisfying

$$\int_1^{+\infty} \frac{1}{y\rho(y)}\,\mathrm{d}y = +\infty, \qquad \sup_{r\geqslant 0}\left|\frac{\langle r\rangle^k D_r^k \rho(r)}{\rho(r)}\right| < +\infty, \quad k \in \mathbb{Z}_+, \tag{2}$$

and for every $\ell \in \mathbb{Z}_+$, $\alpha \in \mathbb{Z}_+^n$, $\beta \in \mathbb{Z}_+^n$ one can find $C = C_{T\alpha\beta} > 0$ such that

$$\max_{j=1,\dots,m}\left|D_t^\ell D_x^\alpha D_\xi^\beta \lambda_j(t,x,\xi)\right| \leqslant C\langle x\rangle^{1-|\alpha|}\rho\big(\langle x\rangle\big)|\xi|^{1-|\beta|}, \quad t \in [-T,T],\ x \in \mathbb{R}^n,\ \xi \in \mathbb{R}^n \setminus 0, \tag{3}$$

$$\left\|D_t^\ell D_x^\alpha D_\xi^\beta b(t,x,\xi)\right\| \leqslant C\langle x\rangle^{-|\alpha|}\rho\big(\langle x\rangle\big)\langle\xi\rangle^{-|\beta|}, \quad t \in [-T,T],\ x \in \mathbb{R}^n,\ \xi \in \mathbb{R}^n, \tag{4}$$

for all $t \in [-T,T]$, $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^n \setminus 0$.

Remark 1. If $\rho = 1$, the estimates (3), (4) coincide with the ones for the global p.d.o. introduced by H.O. Cordes [2] of order $(1,1)$ (see also [3,5,6,8]). Typical weight functions ρ are given (as for the original Wintner condition) by $(\ln|x|)^\sigma$, $\ln|x|(\ln(\ln|x|))^\sigma$, $\ln|x|\ln(\ln|x|)\cdots(\ln(\dots(\ln|x|)\cdots))^\sigma$, $|x| \gg 1$, for $\sigma \in]0,1]$.

(H3) (reduction to a diagonal principal part) We suppose that there exists $S_0(t,x,\xi)$ satisfying the estimates (3) and its inverse $S_0^{-1}(t,x,\xi)$ being of order 0 such that $S_0^{-1}(t,x,\xi)A(t,x,\xi)S(t,x,\xi) = \Lambda(t,x,\xi) = \mathrm{diag}\{\lambda_1,\dots,\lambda_m\}$.

(H4) (functional frames).

(I) (time-dependent weighted spaces) For every $T > 0$, $s \in \mathbb{Z}_+$ one can find $\mathcal{W}(t,z) = \mathcal{W}_{T,s}(t,z) \geqslant 0$, $z \in \mathbb{R}^n$, such that $\mathcal{W}(0,z) = 0$ for $z \geqslant 0$, for every $t \neq 0$ the function $\mathcal{W}(t,z)$ satisfies

$$\Re\big(\big(-\mathcal{W}_t\big(t,\langle x\rangle\big) + \mathrm{i}A*(t,x,D_x) + b(t,x,D_x)\big)u^{\mathcal{W}}, u^{\mathcal{W}}\big)_{H^s(\mathbb{R}^n)} \leqslant 0, \quad |t| \leqslant T, \tag{5}$$

for $u \in CH^{s,\mathcal{W}}(\mathbb{R}^n)$, namely $u^{\mathcal{W}}(t,x) := \mathrm{e}^{-\mathcal{W}(t,\langle x\rangle)}u(t,x) \in C([-T,T] : H^s(\mathbb{R}^n))$, with $H^s(\mathbb{R}^n) = H_2^s(\mathbb{R}^n)$ standing for the $L^2(\mathbb{R}^n)$ Sobolev space of order $s \in \mathbb{Z}_+$, $(f,g)_{H^s(\mathbb{R}^n)} := \sum_{|\alpha|\leqslant s}\int_{\mathbb{R}^n}\partial_x^\alpha f(x)\overline{\partial_x^\alpha g(x)}\,\mathrm{d}x$.

(II) (time-independent weighted spaces). Let $\varkappa \in C^\infty(\mathbb{R}^n : \mathbb{R})$ be a positive function. We define the weighted space $L^2_\varkappa(\mathbb{R}^n)$ by the norm $\|u\|_{L^2_\varkappa(\mathbb{R}^n)} = \|\varkappa u\|_{L^2(\mathbb{R}^n)}$.

We state the first main result:

Theorem 1.1. *Suppose first that the system is of linear partial differential equations. Then the Cauchy problem* (1) *is globally well-posed in* $CH_T^{s,\mathcal{W}}(\mathbb{R}^n)$ *and for all* $T > 0$ *one can find* $C = C_T > 0$ *such that*

$$\sup_{|t|\leqslant T}\left\|u^{\mathcal{W}}(t,\cdot)\right\|_{H^s} \leqslant C_T\left(\left\|u^0\right\|_{H^s(\mathbb{R}^n)} + \int_0^T \sup_{|\tau|\leqslant r}\left\|u^{\mathcal{W}}(\tau,\cdot)\right\|_{H^s}\mathrm{d}r\right). \tag{6}$$

Next, we consider the case of pseudodifferential operators. We assume that

$$\sup_{y\in\mathbb{R}^n}\frac{\rho(y)}{\ln(2+|y|)} < +\infty. \tag{7}$$

Then (6) *holds. Moreover, we get a global well-posedness in the Schwartz class* $\mathcal{S}(\mathbb{R}^n)$*, namely,* $u^0 \in \mathcal{S}(\mathbb{R}^n)$ *and* $f \in C(\mathbb{R} : \mathcal{S}(\mathbb{R}^n))$ *imply* $u \in C(\mathbb{R} : \mathcal{S}(\mathbb{R}^n))$.

Remark 2. The importance of (7) is twofolded. First, for differential equations it is sharp in order to have global well-posedness in the Schwartz class. Indeed, if we consider the scalar case 1D example $\partial_t u + a(x)\partial_x u = 0$, with $a \in C^\infty(\mathbb{R})$, $a = 0$ for $x \leqslant 0$, $a(x) = cx\ln x(\ln\ln x)^\varepsilon$, $x \geqslant 10$, where $\pm c > 0$, $\varepsilon \in]0,1[$, then the Wintner condition holds while (7) fails. Choose $u_0 \in C^\infty(\mathbb{R})$, $u_0 = 0$ for $x \leqslant 10$, $u(x) = \mathrm{e}^{-N(x)\ln x}$, $x \geqslant 10$, where $N(x) = \ln\ln x$. Then $u_0 \in S(\mathbb{R})$ while for every t satisfying $\pm t > 0$ the solution $u(t,x)$ satisfies $\lim_{x\to+\infty}|u(t,x)| = 1$, i.e., $u(t,\cdot) \notin \mathcal{S}(\mathbb{R})$. Secondly, the restriction (7) for the pseudodifferential case in instrumental in the proof of global compositions of pseudodifferential operators with $\mathrm{e}^{\mathcal{W}(t,\langle x\rangle)}$. We can relax a little bit (7) if the pseudodifferential operators are globally analytic-Gevrey with respect to the dual variables ξ, namely, allowing $\rho(y) \sim \ln y(\ln(\ln y))$. The investigations become more involved and it will be done in another work.

We outline also an alternative approach for reduction to hyperbolic systems of the Cordes type or with bounded symbols (cf. D. Gourdin [4]) by means of global diffeomorphisms of $\mathbb{R}^n$ and Liouville type multidimensional changes for hyperbolic systems.

Theorem 1.2. *There exists a global diffeomorphism $\mathbb{R}^n \ni x \mapsto y = \varphi(x) \in \mathbb{R}^n$, with the inverse $\mathbb{R}^n \ni y \mapsto x = \psi(y) = \varphi^{-1}(y) \in \mathbb{R}^n$ such that $\tilde{A}(t, y, D_y) := (\psi^* A)(t, y, D_y)$ is a system with symbol having, at most, a linear growth condition, i.e., we can take $\rho \equiv 1$ (respectively, bounded symbols as in D. Gourdin [4]). Assume that $b = 0$. Then for every $T > 0$ we can find $C > 0$ such that the Cauchy problem admits a unique global solution $u(t, x)$ satisfying*

$$\left\|u(t,\cdot)\right\|_{L^2_\varphi} \leqslant \mathrm{e}^{C|t|}\left\|u^0(\cdot)\right\|_{L^2;\varphi} + \left|\int_0^t \left\|f(\tau,\cdot)\right\|_{L^2;\varphi}\,\mathrm{d}\tau\right|, \quad |t| \leqslant T. \tag{8}$$

Finally, if the system is of partial differential equations, it verifies the finite propagation speed and the domain of dependence properties and the corresponding Cauchy problem is globally well posed in $C^\infty(\mathbb{R}_t \times \mathbb{R}^n)$.

Remark 3. Our finite propagation speed and finite domain of dependence results for systems with unbounded coefficients are more precise in comparison with the typical theorems in [4]. In fact, broadly speaking, we obtain a pull back of a convex cone by a global diffeomorphism with superlinear growth on infinity. We also observe that the Liouville type global change of the space variables above yields a machinery for getting novel results even for hyperbolic systems with unbounded coefficients. It will be shown in another paper.

2. Proofs and examples

The proof of (6) will follow from energy estimates in the weighted spaces. For the sake of simplicity we consider systems of differential equations and $t \in [0, T]$. We have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\left\|u^{\mathcal{W}}(t,\cdot)\right\|^2_{L^2}\right) \\
&\quad = \Re\big(\big(-\mathcal{W}_t\big(t,\langle\cdot\rangle\big) + \Lambda(t,\cdot)\partial_x + b(t,\cdot,\cdot)\big)u^{\mathcal{W}}(t,\cdot), u^{\mathcal{W}}(t,\cdot)\big)_{L^2} + \Re\big(f(t,\cdot), u^{\mathcal{W}}(t,\cdot)\big)_{L^2} \\
&\quad = \Re\big(\big(-\mathcal{W}_t(t,\cdot) - \Lambda(t,\cdot)\partial_x\mathcal{W}(t,\cdot) - \nabla_x\Lambda(t,\cdot) + b\big)u^{\mathcal{W}}(t,\cdot), u^{\mathcal{W}}(t,\cdot)\big)_{L^2} + \Re\big(f(t,\cdot), u^{\mathcal{W},\mathcal{M}}(t,\cdot)\big)_{L^2}.
\end{aligned} \tag{9}$$

It is enough to solve the differential inequalities $\partial_t\mathcal{W}(t,\cdot) - \lambda_j(t,\cdot)\partial_x\mathcal{W}(t,\cdot) \geqslant |\nabla_x\lambda_j(t,x)| + \|b(t,x)\|$, for $x \in \mathbb{R}^n$, $t \in [-T, T]$. We choose $\mathcal{W}(t, y) = (\mathrm{e}^{\mu t} - 1)\theta(\langle y\rangle)$, $y \in \mathbb{R}^n$, $|t| \leqslant T$, with $\theta(r) = \mathrm{e}^{\int 1/(r\rho(r)\,\mathrm{d}r}$. Subtle differential inequalities arguments, taking into account (2)–(4), lead to the validity of (5), with $\mu \gg 1$, depending on T. The proof in the pseudodifferential case relies on nontrivial estimates of commutators in weighted spaces. The details will be given in another work.

Next, we sketch the proof of Theorem 1.2. For the sake of simplicity we exhibit the arguments for $n = 1$. We construct the global diffeomorphism $y = y(x) = \varphi(x)$ for $N \leqslant \pm x < +\infty$ by

$$y(x) = \pm\mathrm{e}^{\pm\int_{\pm N}^{\pm x}\frac{1}{\langle t\rangle\rho(\langle t\rangle)}\,\mathrm{d}t} \quad \left(\text{respectively, } y(x) = \pm\int_{\pm N}^{x}\frac{1}{\langle t\rangle\rho(\langle t\rangle)}\,\mathrm{d}t\right), \qquad \pm x \geqslant N. \tag{10}$$

We observe that $y'(x) = \frac{|y(x)|}{\langle x\rangle\rho(\langle x\rangle)} > 0$ (respectively, $y'(x) = \frac{1}{\langle x\rangle\rho(\langle x\rangle)} > 0$), $y(-x) = -y(x)$, $y'(-x) = y'(x)$, for $x \in]-\infty, -N] \cup [N, +\infty[$, and $\lim_{x\to\pm\infty} y(x) = \pm\infty$. We extend smoothly $y(x)$ in $[-N, N]$ by demonstrating a global version of the Borel lemma. More precisely, we define $y(x) = \int_0^x w(t)\,\mathrm{d}t$ for $x \in [-N, N]$, where

$$w(x) = y'(N) + \sum_{j=1}^{\infty}\frac{s_j}{j!}\big(x^2 - N^2\big)^j\theta\big(\big(N^2 - x^2\big)C_j\big), \tag{11}$$

with $\theta \in C_0^\infty(\mathbb{R})$, $0 \leqslant \theta \leqslant 1$, $\theta(z) = 1$ for $|z| \leqslant 1/2$, $\operatorname{supp}\theta \subset [-1, 1]$, $C_j \nearrow +\infty$, and s_j are chosen (determined uniquely) by the requirement that $w(x)$ has the same Taylor expansion at $x = N$ (respectively, $x = -N$) as $y'(x)$, y defined by (10), at $x = N$ (respectively, $x = -N$). By letting $C_j \gg 1$ for $j \in \mathbb{N}$ we guarantee $w(x) \geqslant y(N)/2$ for

$x \in [-N, N]$, which allows us to show that $y(x)$ becomes diffeomorphism of $\mathbb{R}$. The proof in the multidimensional case is more involved. The key idea is to use the polar coordinates and carry out the one-dimensional construction for the polar radius $\rho = \|x\|$ away from the origin while for $\|x\| \leqslant N$ the generalization of the Borel relies on subtle technical arguments.

By the change of the variables $x \mapsto y$ we reduce the original Cauchy problem to

$$\partial_t v(t, y) + \mathrm{i}\tilde{A}(t, y, D_y)v + \tilde{b}(t, y)v = \tilde{f}(t, y), \quad v(0, y) = v^0(y), \quad y \in \mathbb{R}^n, t \in \mathbb{R}, \tag{12}$$

where $v(t, y) = u(t, \varphi(y))$, $t \in \mathbb{R}$, $y \in \mathbb{R}^n$, $v(0, y) = v^0(y) := u^0(\varphi(y))$. Since the new system satisfies the linear growth hypotheses in [2] (respectively, the boundedness conditions, e.g., see [4]) for the principal symbol, we apply the well known estimates in the y coordinates. We complete the proof by returning to the x variables and obtain that $\varkappa(x) = \sqrt{|J_\varphi(x)|}$, where $J_\varphi(x)$ stands for the determinant of the Jacobian matrix $\nabla\varphi(x)$.

As it concerns the finite propagation and the domain of dependence we stress that the arguments used in [9,1,4] are not applicable in our case. The proof follows from the reduction to hyperbolic systems with bounded coefficients by global diffeomorphism of $\mathbb{R}^n$ and then we use the result in [4].

Remark 4. Note that in the original coordinates, even for the autonomous case $A = A(x, D_x)$, we have not the finite propagation speed for all $t > 0$, but only for every fixed $T > 0$ with the cone depending on the domain if the coefficients are not bounded. In fact, our construction of the global diffeomorphism implies more precise estimates for the propagation. Consider the second order wave equation $\partial_t^2 u - (1 + x^2)\rho^2(x^2)\Delta u = 0$. We can transform to a second order strictly hyperbolic equations with bounded characteristics. The finite propagation speed cone for $|t| \leqslant T$ in the y variables for the ball B_R is given by $\Gamma_R^T := \{\|y\| \leqslant R + \lambda_0|t|\}$, for some $\lambda_0 > 0$, and it is transformed in the original x variables in $\widetilde{\Gamma_R^T} := \{\|x\| \leqslant \varkappa^{-1}(R + \lambda_0|t|)\}$, which is not convex and gives better estimates of the support of $u(t, x)$ in comparison with the convex hull of $\widetilde{\Gamma_R^T}$.

We outline some patterns of phenomena for the scalar 1-D Cauchy problem $\partial_t u + a(x)\partial_x u = 0$, $u(0, x) = u^0(x) \in \mathcal{S}(\mathbb{R})$, where $a(x) \in C^\infty(\mathbb{R} : \mathbb{R})$, and $a(x) = C_\pm x\rho(|x|)$ for $\pm x > N$, where $N \gg 1$, $C_+ \neq 0$, $C_- \neq 0$, with ρ satisfying the Wintner condition (2), but not (7). Then the Cauchy problem is globally well-posed in $C(\mathbb{R} : C^\infty(\mathbb{R}))$ but not well-posed in $C(]-T, T[: \mathcal{S}(\mathbb{R}))$ for all $T > 0$. More precisely, if $C_+C_- < 0$, then the following blow-up occurs, there exists $k > 0$ such that $\|\langle x\rangle^{-k}u(t, \cdot)\|_{L^2} = +\infty$, for infinitely many $t \neq 0$. Finally, we exhibit a "weak dissipative" type behavior for positive (respectively, negative) times assuming both C_+, C_- positive (respectively, negative), namely: the blow-up holds for all $t < 0$ (respectively, $t > 0$) while the Cauchy problem is globally well-posed in $C([0, +\infty[: \mathcal{S}(\mathbb{R}))$ (respectively, $C(]-\infty, 0] : \mathcal{S}(\mathbb{R}))$. More generally, the global well-posedness in $\mathcal{S}(\mathbb{R})$ holds for $t > 0$ (respectively, $t > 0$) provided $a'(x) \geqslant 0$ (respectively, $a'(x) \leqslant 0$), but not for $t \in \mathbb{R}$.

3. Final remarks

Our examples in the previous section imply that the choice of the weight function $\mathcal{W}$ is optimal under the Wintner condition. We can also construct a representation of the solutions by means of global FIO of the type $\int_{\mathbb{R}^n} \exp(\mathrm{i}\varphi(t, x, \xi))q(t, x, \xi)\widehat{u}^0(\xi)\,\bar{\mathrm{d}}\xi$, where the phase function φ does not obey, in the superlinear case, the conditions introduced by M. Ruzhansky and M. Sugimoto [10]. As an outcome of our constructions we are able to provide explicit examples of FIO which do not satisfy global $L^2(\mathbb{R}^n)$ estimates in [10]. Indeed, take $q \equiv 1$, $u_0 \in \mathcal{S}(\mathbb{R})$ as above, while $\varphi(t, x, \xi) = x^{e^{ct}}\xi$ for $|x| > C|t|$, $C \gg 1$. The FIO does not act continuously in $L^2(\mathbb{R})$ for $t \neq 0$. It follows from the explicit constructions and the domain of dependence for solutions of $\partial_t u + c_\pm x \ln|x|\partial_x u = 0$ for $|x| \gg 1$. More details will be given in another work.

References

[1] J. Chazarain, A. Piriou, Introduction to the Theory of Linear Partial Differential Equations, Studies in Mathematics and its Applications, vol. 14, North-Holland Publishing Co., Amsterdam–New York, 1982.

[2] H.O. Cordes, The Technique of Pseudodifferential Operators, LMS Lecture Note Series, vol. 202, Cambridge Univ. Press, Cambridge, 1995.

[3] S. Coriasco, L. Rodino, Cauchy problem for SG-hyperbolic equations with constant multiplicities, Ricerche Mat. 48 (suppl.) (1999) 25–43.

[4] D. Gourdin, Problème de Cauchy non caractéristique pour les systèmes hyperboliques à caractéristiques de multiplicité variable, Domaine de dépendence, Comm. Partial Differential Equations 4 (1979) 447–507.

[5] D. Gourdin, T. Gramchev, Global in time solutions of evolution equations in scales of Banach function spaces in $\mathbb{R}^n$, Bull. Sci. Math. (2007).
[6] D. Gourdin, M. Mechab, Solutions globales d'un problème de Cauchy linèaire, J. Funct. Anal. 202 (1) (2003) 123–146.
[7] P. Hartman, Ordinary Differential Equations, John Wiley & Sons, New York–London–Sydney, 1964.
[8] I. Kamotski, M. Ruzhansky, Representation of solutions and regularity properties for weakly hyperbolic systems, in: Pseudo-Differential Operators and Related Topics, in: Oper. Theory Adv. Appl., vol. 164, Birkhäuser, Basel, 2006, pp. 53–63.
[9] S. Mizohata, On the Cauchy Problem, Notes and Reports in Mathematics in Science and Engineering, vol. 3, Science Press and Academic Press, 1985.
[10] M. Ruzhansky, M. Sugimoto, Global L^2-boundedness theorems for a class of Fourier integral operators, Comm. Partial Differential Equations 31 (2006) 547–569.
[11] A. Wintner, The nonlocal existence problem of ordinary differential equations, Amer. J. Math. 67 (1945) 277–284, 125–132.