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Differential Geometry

Para-Kähler Einstein metrics on homogeneous manifolds ☆

Dimitri V. Alekseevsky [a], Costantino Medori [b], Adriano Tomassini [b]

[a] *The University of Edinburgh, James Clerk Maxwell Building, The King's Buildings, Mayfield Road, Edinburgh, EH9 3JZ, UK*
[b] *Dipartimento di Matematica, Università di Parma, Viale G.P. Usberti, 53/A, 43100 Parma, Italy*

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Abstract

A para-Kähler structure on a manifold M is a pair (g, K) where g is a pseudo-Riemannian metric and K is a parallel field of skew-symmetric endomorphisms with $K^2 = \mathrm{Id}$. We give a description of all invariant para-Kähler structures (g, K) on homogeneous manifolds $M = G/H$ of semisimple Lie groups G. ***To cite this article: D.V. Alekseevsky et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Metriques para-Kähler Einstein sur des variétés homogènes. Une structure para-Kählérienne sur une variété M est la donnée d'un paire (g, K), où g est une metrique pseudo-riemannienne et K est un champ parallel d'endomorphismes anti-symétriques qui satisfait $K^2 = \mathrm{Id}$. On donne une description de toutes les structures para-Kählériennes invariantes (g, K) sur des variétés homogènes $M = G/H$, où G est un group de Lie semisimple. ***Pour citer cet article : D.V. Alekseevsky et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

1. Para-Kähler manifolds

An *almost paracomplex structure* on a $2n$-dimensional manifold M is a field K of endomorphisms of the tangent bundle TM such that $K^2 = \mathrm{Id}_{TM}$ and the two eigendistributions $T^{\pm}M := \ker(\mathrm{Id} \mp K)$ have the same rank. Such structures were introduced by Libermann in [9]. For a survey on paracomplex geometry see e.g. [6]. An almost paracomplex structure K is said to be *integrable* if the distributions $T^{\pm}M$ are involutive. This is equivalent to the vanishing of the *Nijenhuis tensor* N_K defined by

$$N_K(X, Y) = [X, Y] + [KX, KY] - K[KX, Y] - K[X, KY], \tag{1}$$

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E-mail addresses: D.Aleksee@ed.ac.uk (D.V. Alekseevsky), costantino.medori@unipr.it (C. Medori), adriano.tomassini@unipr.it (A. Tomassini).

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for vector fields X, Y on M (see e.g. [5]). In such a case K is called a *paracomplex structure* and (M, K) a *paracomplex manifold*.

A map $f:(M, K) \to (M', K')$ between two paracomplex manifolds is said to be *para-holomorphic* if $\mathrm{d}f \circ K = K' \circ \mathrm{d}f$.

The module $C^n = \{(z^1, \dots, z^n) \mid z^j \in C\}$ over the algebra $C = \{z = x + ey \mid x, y \in \mathbb{R}, e^2 = 1\}$ of paracomplex numbers has a natural paracomplex structure K given by the multiplication by e. A paracomplex manifold (M, K) admits an atlas of para-holomorphic coordinates, i.e. such that the transition functions are para-holomorphic. As in the complex case, the bundle $\Lambda^r(T^*M \otimes C)$ of C-valued r-form is decomposed into a direct sum of (p, q)-forms $\Lambda^{p,q}M$ and, according to the above decomposition, the exterior differential d is represented as a direct sum $d = \partial + \bar{\partial}$. Moreover, a paracomplex version of the Dolbeault Lemma holds (see [5]).

A *para-Kähler manifold* is the datum of (M, g, K), where g is a pseudo-Riemannian metric and K is a skew-symmetric paracomplex structure which is parallel with respect to the Levi-Civita connection of g. In such a case the pair (g, K) gives rise to a symplectic form ω on M defined by $\omega(\cdot, \cdot) = g(K\cdot, \cdot)$.

A para-Kähler manifold is equivalent to the datum of a symplectic manifold (M, ω) together with two complementary involutive Lagrangian distributions $T^{\pm}M$ (see [4]). It turns out that the Ricci form of a para-Kähler metric g is a closed $(1, 1)$-form which can be locally represented by

$$\rho = e\partial\bar{\partial}\log\bigl(\det(g_{\alpha\bar{\beta}})\bigr). \tag{2}$$

Let (M, K, vol) be an oriented manifold with a paracomplex structure K and a (real) volume form vol. In local para-holomorphic coordinates $z = (z^1, \dots, z^n)$ we can write $vol = V(z, \bar{z})\,\mathrm{d}z^1 \wedge \mathrm{d}\bar{z}^1 \wedge \cdots \wedge \mathrm{d}z^n \wedge \mathrm{d}\bar{z}^n$. Then

$$\kappa = e\partial\bar{\partial}\log\bigl((-e)^n V\bigr) \tag{3}$$

defines a real global closed 2-form of type $(1, 1)$ on the oriented paracomplex manifold (M, K, vol), which is called the *canonical form* on (M, K, vol). We have the following:

Proposition 1. *Let (M, K, ω, g) be an oriented para-Kähler manifold and vol^g the volume form associated with the metric g. Then the Ricci form ρ of the para-Kähler manifold M coincides with the canonical form κ of (M, K, vol^g). In particular ρ depends only on the paracomplex structure and the volume form.*

2. Homogeneous para-Kähler Einstein manifolds

Let $M = G/H$ be a homogeneous reductive manifold with an invariant volume form vol. We fix a reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ of the Lie algebra $\mathfrak{g}$ and identify the subspace $\mathfrak{m}$ with the tangent space T_oM at the point $o = eH$.

Now we assume that G/H is endowed with an invariant paracomplex structure K and we get a formula for the canonical form κ. We extend the endomorphism $K|_{T_oM} = K|_{\mathfrak{m}}$ to the $\mathrm{Ad}_G(H)$-invariant endomorphism $\tilde{K}$ of $\mathfrak{g}$ with kernel $\mathfrak{h}$ and we denote by the same symbol the associated left-invariant field of endomorphisms on the group G. The following proposition is a version for paracomplex manifolds of a result due to Koszul (see [8, Thm. 1]):

Proposition 2. *Let $M = G/H$ be a homogeneous manifold with an invariant volume form vol and an invariant paracomplex structure K. Then the pull-back $\pi^*\kappa$ to G of the canonical 2-form κ associated with (vol, K) at the point $o = eH$ is given by*

$$2(\pi^*\kappa)_e = \mathrm{d}\psi, \tag{4}$$

where ψ is the $\mathrm{ad}_{\mathfrak{h}}$-invariant 1-form on $\mathfrak{g}$ given by

$$\psi(X) = -\operatorname{tr}_{\mathfrak{g}/\mathfrak{h}}(\mathrm{ad}_{\tilde{K}X} - \tilde{K}\,\mathrm{ad}_X), \quad \forall X \in \mathfrak{g}. \tag{5}$$

The 1-form ψ (and the associated left-invariant 1-form on G) is called the *Koszul form*.

Invariant almost paracomplex structures K are defined by an $\mathrm{Ad}_G(H)$-invariant decomposition $\mathfrak{m} = \mathfrak{m}^+ + \mathfrak{m}^-$ where $\mathfrak{m}^+$ and $\mathfrak{m}^-$ are vector subspaces of the same dimension such that $\mathfrak{h} + \mathfrak{m}^{\pm}$ are subalgebras.

Invariant symplectic structures on $\mathfrak{m}$ are defined by a closed $\mathrm{Ad}_G(H)$-invariant 2-form ω on $\mathfrak{g}$ with kernel $\mathfrak{h}$. It can be checked that an invariant symplectic form ω is of type $(1,1)$ with respect to an invariant paracomplex structure K if and only if $\omega|_{\mathfrak{m}^\pm} = 0$.

Proposition 3. *Invariant para-Kähler structures on a reductive homogeneous manifold $M = G/H$ are defined by triples $(\omega, \mathfrak{m}^+, \mathfrak{m}^-)$, where $\mathfrak{m} = \mathfrak{m}^+ + \mathfrak{m}^-$ is an $\mathrm{Ad}_G(H)$-invariant decomposition such that $\mathfrak{h} + \mathfrak{m}^\pm$ is a subalgebra of $\mathfrak{g}$ and ω is a closed $\mathrm{Ad}_G(H)$-invariant 2-form on $\mathfrak{g}$ with kernel $\mathfrak{h}$ such that $\omega|_{\mathfrak{m}^\pm} = 0$.*

Note that an invariant volume form on a homogeneous manifold $M = G/H$ exists if and only if the isotropy representation $j(H)$ is unimodular (i.e. $\det(j(h)) = 1$ for all $h \in H$), and it is defined up to a constant scaling. By Propositions 2 and 3 we get the following:

Theorem 4. *Let $(M = G/H, K)$ be a homogeneous paracomplex manifold with an invariant volume form vol. Then any invariant para-Kähler structure (K, ω) has the same Ricci form ρ which is the canonical form h of (M, K, vol) and there exist invariant para-Kähler Einstein structures with non-zero scalar curvature if and only if the canonical form ρ is non-degenerate. These structures are given by pairs (K, ω), with $\omega = \lambda\rho$, $\lambda \neq 0$.*

3. Homogeneous para-Kähler Einstein manifolds of a semisimple group

The aim of this section is to describe invariant para-Kähler Einstein structures on homogeneous manifolds $M = G/H$ of semisimple groups G. Since the Killing form $B|_{\mathfrak{h}}$ is non-degenerate, the B-orthogonal complement $\mathfrak{m} = \mathfrak{h}^\perp$ of $\mathfrak{h}$ defines a reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$.

We need the following important result (see [7, §3]):

Theorem 5. *A homogeneous manifold $M = G/H$ of a semisimple Lie group G admits an invariant para-Kähler structure (K, ω) if and only if it is a covering of a semisimple adjoint orbit $\mathrm{Ad}_G(h) = G/Z_G(h)$ that is the adjoint orbit of a semisimple element $h \in \mathfrak{g}$.*

Note that in this case $Z_G^0(h) \subset H \subset Z_G(h)$, where $Z_G^0(h)$ denotes the connected centralizer of h in G and the element h is $\mathfrak{h}$-*regular*, i.e. its centralizer in $\mathfrak{g}$ is $Z_{\mathfrak{g}}(h) = \mathfrak{h}$.

We recall that a gradation $\mathfrak{g} = \mathfrak{g}_{-k} + \cdots + \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1 + \cdots + \mathfrak{g}_k$, $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ of a semisimple Lie algebra $\mathfrak{g}$ is called *fundamental*, if the subalgebras $\mathfrak{m}_\pm = \sum_{i>0} \mathfrak{g}_i$ are generated by $\mathfrak{g}_{\pm 1}$, respectively. All fundamental gradations of a real semisimple Lie algebra are easily described in terms of the corresponding Satake diagram. We have the following (see [1, Prop. 3.7]):

Proposition 6. *There is a natural 1–1 correspondence between invariant paracomplex structures K on a homogeneous manifold G/H of a semisimple Lie group G which is a covering of a semisimple adjoint orbit $M = \mathrm{Ad}_G(h)$ and $\mathrm{Ad}_G(H)$-invariant fundamental gradations of the Lie algebra $\mathfrak{g}$ with $\mathfrak{g}_0 = \mathfrak{h}$. The gradation defines the paracomplex structure K such that $K|_{\mathfrak{m}_\pm} = \pm\mathrm{Id}$.*

Let $\mathfrak{h} = Z_{\mathfrak{g}}(h)$ be the centralizer of a semisimple element h. We have a B-orthogonal decomposition $\mathfrak{h} = \mathfrak{z} + \mathfrak{h}'$, where $\mathfrak{z}$ is the center of $\mathfrak{h}$ and $\mathfrak{h}' = [\mathfrak{h}, \mathfrak{h}]$. We need the following known proposition:

Proposition 7. *Let $M = G/H$ be a homogeneous manifold as in the previous proposition. Then there exists a natural 1–1 correspondence between $\mathrm{Ad}_G(H)$-invariant elements $z \in \mathfrak{z}$ and closed invariant 2-forms ω_z on $M = G/H$ given by*

$$\omega_z(X, Y) = B\big(z, [X, Y]\big), \quad \forall X, Y \in \mathfrak{m} = T_o M \subset \mathfrak{g}. \tag{6}$$

Moreover ω_z is a symplectic form if and only if z is $\mathfrak{h}$-regular.

In view of Proposition 7 we obtain the following:

Corollary 8. *Any invariant paracomplex structure K on $M = G/H$ is skew-symmetric with respect to any invariant symplectic structure ω. Hence, (K, ω) gives rise to an invariant para-Kähler structure.*

Now we can state our main result which gives a description of homogeneous para-Kähler Einstein manifolds over a semisimple Lie group G, analogue to the description of compact homogeneous Kähler–Einstein manifolds given in [3].

Theorem 9. *Let $M = G/H$ be a homogeneous manifold of a semisimple Lie group G which admits an invariant para-Kähler structure and K be the invariant para-complex structure on M. Then the push-down of the differential $\mathrm{d}\psi$ of the Koszul 1-form ψ on G gives rise to a canonical invariant symplectic structure ρ on M such that $g_{\lambda,K} := \lambda^{-1}\rho \circ K$ is an invariant para-Kähler Einstein metric on M with Einstein constant $\lambda \neq 0$. This construction gives all invariant para-Kähler–Einstein metrics on M.*

The results contained in Theorems 4 and 9 are proved in [2], where the explicit expression for the Kähler metric $g_{\lambda,K}$ is provided and some examples are considered.

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