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Number Theory

Rank of elliptic surfaces and base change ☆

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Abstract

We study the variations of the rank of fibers of an elliptic surface with minimal model over k isomorphic to $\mathbb{P}^2_k$. We show that an infinite number of fibers have rank at least the generic rank plus two. ***To cite this article: C. Salgado, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Rang de surfaces elliptiques et changements de base. On étudie les variations du rang des fibres dans une surface elliptique. On montre que si son modèle minimal est $\mathbb{P}^2_k$ alors il existe une infinité de fibres avec un rang égal au moins au rang générique augmenté de deux unités. ***Pour citer cet article : C. Salgado, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

1. Introduction

Throughout this Note k will be a number field. Let B be a smooth projective integral curve and $\pi : E \to B$ be an elliptic surface with a section defined over k [7, Ch. 3]. We shall be interested in comparing the generic rank $\mathrm{rk}E(k(B))$ (the usual Mordell–Weil theorem holds in this generality, see [7, Ch. III, Thm. 6.1]) with the Mordell–Weil rank of the fibers E_t, where $t \in B(k)$.

Our object of interest are elliptic surfaces whose function field is isomorphic to $k(\mathbb{P}^2)$ and whose minimal model over k is isomorphic to $\mathbb{P}^2_k$. Other possibilities for the minimal model of a k-rational elliptic surface are del Pezzo surfaces of degree five or six. These cases will be treated by the author in her Ph.D. thesis.

Theorem 1.1. *Let $E \to B$ be a k-rational elliptic surface with minimal model isomorphic to $\mathbb{P}^2_k$. Then there exists a finite and dominant morphism $C \to B$ where C is a curve of genus $\leqslant 1$ such that $C(k)$ is infinite and the base changed elliptic surface $E \times_B C \to C$ satisfies*:

$$\mathrm{rk}(E \times_B C)\big(k(C)\big) \geqslant \mathrm{rk}E\big(k(B)\big) + 2.$$

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Since the curve in the theorem has infinitely many rational points we have the immediate consequence after applying a theorem of Silverman on specializations [8].

Corollary 1.2. *Let E be as in the theorem. There exists infinitely many points $t \in B(k)$ such that*

$$\mathrm{rk} E_t(k) \geqslant \mathrm{rk} E\big(k(B)\big) + 2.$$

Remark 1. Corollary 1.2 generalizes in several ways Theorem C of [1]; the lower bound is larger and the proof is valid over any number field.

Remark 2. The class of k-rational elliptic surfaces with minimal model $\mathbb{P}^2_k$ is actually very big. Namely all rational surfaces obtained by blowing up the base locus of $\{tF + uG;\ [t,u] \in \mathbb{P}^1\}$ where F (smooth) and G are homogeneous cubic polynomials in $k[x,y,z]$. Also one can show that all the k-rational elliptic surfaces with bad fibers of type II^* or III^* are in this class. For a list of possible bad fibers see [4, I.4.1].

1.1. New and old sections

Let $\pi : E \to B$ an elliptic surface (not necessarily rational). Given an integral curve $\iota : C' \hookrightarrow E$ which is not contained in a fiber of π, if $\nu : C \to C'$ is its normalization we obtain a new elliptic surface $\pi_C : E_C := E \times_B C \to C$ via the morphism $f = \pi \circ \iota \circ \nu : C \to B$. Each section $\sigma : B \to E$ induces naturally a section $(\sigma, \mathrm{id}) : C = B \times_B C \to E \times_B C$. These will be called *old sections*. The elliptic surface E_C inherits a *new section* given by $\sigma_C^{\mathrm{new}} = (\iota \circ \nu, \mathrm{id}) : C \to E \times_B C$.

If C' is not contained in a fiber of π and is not a section of π then the new section is clearly different from any old section induced in the base changed surface, but may be $\mathbb{Z}$-dependent.

The idea—which can be traced back to Néron [5]—behind our proof of Theorem 1.1 is to construct a curve $C' \hookrightarrow E$ such that the new section will be responsible for the augmentation of the rank.

1.2. Notation

The set of sections $\sigma : B \to E$ of the structural morphism $\pi : E \to B$ is denoted by $\mathrm{Sec}(\pi)$, it carries a natural group structure [7, Ch. III, 3.10] essentially because it is isomorphic to the Mordell–Weil group of the generic fiber $E(k(B))$. The multiplication by n is defined only on an open subset of E but this enables us to define the image or pull-back of a curve.

The division of the text is the following. The proof of the theorem is contained in Sections 2 and 4. In Section 2 we prove that given an algebraic family of curves that are not sections nor fibers then all but a finite number of them induce a new independent section. Section 3 gives a geometric description of rational elliptic surfaces that will help us to construct, in Section 4, pencils of curves with the requested properties finishing the proof of the theorem.

2. A key proposition

The main technical tool we will apply to prove the theorem is the following general proposition:

Proposition 2.1. *Let $\pi : E \to \mathbb{P}^1_k$ be an elliptic surface. Let $\mathcal{L} = \{L_u\colon\ u \in \mathbb{P}^1\}$ be a non-constant pencil of curves in E whose generic member is irreducible and neither a section nor a fibre of π. Then, for almost all $u \in \mathbb{P}^1(k)$, the new section $\sigma_{L_u}^{\mathrm{new}}$ is $\mathbb{Z}$-independent of the old sections.*

The most useful criterion to evaluate the independence of a new section from old ones is the following:

Lemma 2.2. *Let $C \hookrightarrow E$ be an irreducible curve in E which is not a component of a fibre of π. Then the new section σ_C^{new} induced by C is independent of the old sections if and only if for every section $\Sigma \hookrightarrow E$ and for every $n \in \mathbb{N}$, the curve C is not a component of $[n]_E^{-1}(\Sigma)$.*

Let us give some indications of how Lemma 2.2 can be used to prove Proposition 2.1. Using Kummer's theory for elliptic curves [2], we can show that there exists an $n_0 \in \mathbb{N}$ such that if L_t is not a component of $\bigcup_{\Sigma \in \mathrm{Sec}(\pi),\, n \leqslant n_0} [n]_E^{-1} \Sigma$, then L_t produces a new section which is independent of the old ones. From the theory of Néron–Tate heights [3,6], it is possible to make further restrictions: there exists a *finite* set of sections $S \subset \mathrm{Sec}(\pi)$ such that if L_t is not a component of $\bigcup_{\Sigma \in S,\, n \leqslant n_0} [n]_E^{-1} \Sigma$, then L_t is not a section. Of course, the set of irreducible curves in the above scheme is finite and this is enough to prove the proposition.

3. Cubic pencils for rational elliptic surfaces

Let F and G be homogeneous cubic polynomials in three variables such that F and G are coprime. Assume that $V(F) \subset \mathbb{P}^2_k$ is smooth. The pencil $\mathcal{P} = \{V(tF + uG) \subset \mathbb{P}^2_k;\ (t:u) \in \mathbb{P}^1\}$ has nine base points (counted with multiplicity) and the blow-up of $\mathbb{P}^2_k$ at these base points defines a smooth rational elliptic surface $\pi : E_{\mathcal{P}} \to \mathbb{P}^1$. In fact this surface is birational to the following (possibly singular) surface,

$$E'_{\mathcal{P}} = \big\{\big[(x:y:z);(t:u)\big] \in \mathbb{P}^2 \times \mathbb{P}^1:\ tF(x,y,z) + uG(x,y,z) = 0\big\}$$

and π is just the composition of the inclusion with the projection $\mathbb{P}^2 \times \mathbb{P}^1 \to \mathbb{P}^1$.

Proposition 3.1. *([4, Lecture 4].) Over an algebraically closed field, any smooth rational elliptic surface with a section is isomorphic to some $E_{\mathcal{P}}$. The choice of $\mathcal{P}$ is non-canonical.*

Over a number field this is valid in general only after a base extension. In the case treated here this is also valid over k since the minimal model over k is $\mathbb{P}^2_k$.

4. Construction of the base change envisaged in Theorem 1.1

Let $\pi : E \to \mathbb{P}^1_k$ be a k-rational elliptic surface as in Theorem 1.1. From Section 3 we obtain a pencil $\mathcal{P}$ such that E is an elliptic surface isomorphic to $E_{\mathcal{P}}$. We will need two base points to perform the base changes. The following lemma tells us when this condition is satisfied:

Lemma 4.1. *If $\mathcal{E}$ has no fiber of type II^* then the pencil $\mathcal{P}$ has at least two distinct base points.*

The proof of this lemma is achieved through case-by-case considerations.

If $\mathcal{E}$ has a fiber of type II^* its generic rank is zero; a quadratic base change ramified over v such that the fiber $\pi^{-1}(v)$ is the II^*-fiber and ramified at another point v' such that $\pi^{-1}(v')$ is a smooth fiber is still a rational elliptic surface $\mathcal{E}' \to B' \simeq \mathbb{P}^1$ for if $e(V)$ is the Euler characteristic of a variety V then since all the singular fibers different from II^* occur twice in the base change $12(p_g(\mathcal{E}') + 1) = 8 + 2\sum_{v \neq t \in \mathbb{P}^1} e(\pi^{-1}(t)) = 12$. Note that the terms in the summation are non-zero only at the singular fibers. The only reducible fiber of the base changed surface is of type IV^*. One can perform this base change such that $\mathrm{rk}(\mathcal{E}'(k(B'))) \geqslant \mathrm{rk}(\mathcal{E}(k(B))) + 1$. Then one can construct a family of rational curves, on its minimal model, defined over k. Applying Proposition 2.1 one has a base changed surface $\mathcal{E}'' \to B''$ such that $\mathrm{rk}(\mathcal{E}''(k(B''))) \geqslant \mathrm{rk}(\mathcal{E}'(k(B')) + 1 \geqslant \mathrm{rk}(\mathcal{E}(k(B))) + 2$.

We will now treat the main case, i.e. when there are at least two distinct base points.

Let $q : E \to \mathbb{P}^2$ denote the blow-up of $\mathbb{P}^2$ at the nine base points of the pencil $\mathcal{P}$. The curve C in the statement of the theorem will be of the form $C_1 \times_B C_2$, where:

(i) Each C_i is the normalization of the strict transform of a rational curve in $\mathbb{P}^2$ passing through some of the base points of the pencil $\mathcal{P}$.
(ii) If ι_j denotes the closed embedding of C_j into E, then $\deg(\pi \circ \iota_j) = 2$. (This will guarantee that $C_1 \times_B C_2$ has genus $\leqslant 1$.)
(iii) The new sections $\sigma_i := \sigma^{\mathrm{new}}_{C_i} : C_i \to E \times_B C_i$ are linearly independent from the old sections.

In general only one base point is k-rational, nevertheless for simplicity we will give the proof when there are two k-rational base points and point out at the end the necessary modifications to carry out the general proof.

Let $\mathcal{L}_1 = \{L_u^1\colon u \in \mathbb{P}^1\}$ and $\mathcal{L}_2 = \{L_u^2\colon u \in \mathbb{P}^1\}$ be two pencils of lines through P_1 and P_2 respectively. The strict transform of L_u^i in E (by q) will be denoted by M_u^i. Let $\pi(t,1) : E(t,1) \to M_t^1$ denote the elliptic surface obtained from E by base-changing via $M_t^1 \to B$. Proposition 2.1 shows that for all $t \in \mathbb{P}^1(k)$ outside a finite set $\Phi_1 \subset \mathbb{P}^1(k)$, the section $\sigma_{M_t^1}^{\text{new}} : M_t^1 \to E(t,1)$ is independent of the old sections. Let N_s^2 denote the pre-image of M_s^2 in $E(t,1)$. By another application of Proposition 2.1, we obtain that for all $s \in \mathbb{P}^1(k)$ outside a finite set $\Phi_2 \subset \mathbb{P}^1(k)$, the new section $\sigma_{N_s^2}^{\text{new}} : N_s^2 \to E(t,1) \times_{M_t^1} N_s^2$ will be independent of the old ones. Therefore, by taking $(u,s) \in \mathbb{P}^1(k) \times \mathbb{P}^1(k) \backslash (\Phi_1 \times \mathbb{P}^1(k) \cup \mathbb{P}^1(k) \times \Phi_2)$ we have constructed two new independent sections.

Finally, to show that infinitely many of the curves $C = M_u^1 \times_B M_s^2$ are curves with the properties envisaged by Theorem 1.1 we have to prove that for an infinite set of $u, s \in \mathbb{P}^1(k)$ $\operatorname{card}(C(k)) = \infty$. For that, let us fix $s \in \mathbb{P}^1$ and consider $Q(u)$ the point in the intersection of L_u^1 and L_s^2. For each u there is a unique cubic $\Gamma_{t(u)} = t(u)F + G$ of the pencil $\mathcal{P}$ that passes through $Q(u)$. This cubic intersects each L^i in three points: $\Gamma_{t(u)} \cap L^i = \{P_i, Q(u), R_i(u)\}$. Let $Q'(u), R_i'(u)$ be the corresponding points in M^i. The four points $(Q'(u), Q'(u))$, $(R_1'(u), Q'(u))$, $(Q'(u), R_2'(u))$ and $(R_1'(u), R_2'(u))$ are $k(u)$-rational and lie in the fiber product $M_u^1 \times_B M_s^2$ for fixed s since $\pi(Q'(u)) = t(u) = \pi(R_i(u))$. Let u_0 be such that L_1 is tangent to $\Gamma_{t(u_0)}$ $(Q(u), R_2(u))$ and $(R_1(u), R_2(u))$ intersect the (one can check that it is non-singular) point $(Q(u_0), R_2(u_0))$. Since torsion sections do not intersect each other in a non-singular point, one of the two sections has infinite order. This elliptic surface has rank at least one thus by Silverman's specialization theorem [8] almost all fibers have rank at least one.

The case of non-trivial Galois action: In the general case we have one k-rational base point and various Galois orbits. We apply the same procedure but instead of taking pencils of lines through one point, we take more general pencils of rational curves. For example if we have two Galois orbits given by $O_1 = \{P_1\}$ and $O_2 = \{P_2, P_3, P_4\}$. We take $\mathcal{L}_1$ the pencil of lines through P_1 and $\mathcal{L}_2$ the pencil of conics through P_1, P_2, P_3, P_4. The generic member of $\mathcal{L}_2$ is defined over k. The rest of the proof goes exactly the same way as for the simpler case since for each curve $L'_{2,u}$ in the pencil of strict transforms $\mathcal{L}_2$ the degree of the morphism $\phi_{2,u} : L'_{2,u} \to B$ is also two.

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References

[1] H. Billard, Sur la répartition des points rationnels de surfaces elliptiques, J. Reine Angew. Math. 505 (1998) 45–71.
[2] M. Hindry, Autour d'une conjecture de Serge Lang, Inven. Math. 94 (1988) 575–603.
[3] S. Lang, Diophantine Geometry, Interscience Tracts in Pure Appl. Math., vol. 11, John Wiley and Sons, 1962.
[4] R. Miranda, The basic theory of elliptic surfaces, Dottorato di Ricerca in Matematica, ETS Editrice, Pisa, 1989.
[5] A. Néron, Propriétés arithmétiques de certaines familles de courbes algébriques, in: Proc. Inter. Congress of Math., vol. III, Amsterdam, 1954, North-Holland Publishing Co., Amsterdam, 1956, pp. 481–488.
[6] T. Shioda, On the Mordell Weil lattices, Comment. Math. Univ. St. Pauli 39 (1990) 211–240.
[7] J. Silverman, Advanced Topics in the Arithmetic of Elliptic Curves, second ed., Grad. Texts in Math., vol. 151, Springer-Verlag, 1999.
[8] J. Silverman, Heights and the specialization map for families of abelian varieties, J. Reine Angew. Math. 342 (1983).