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Optimal Control

Exact controllability of a 3D piezoelectric body

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Abstract

In this Note we study the exact controllability of a three-dimensional body made of a material whose constitutive law introduces an elasticity-electricity coupling. We show that, without any geometrical assumption, two controls (the elastic and the electric controls) acting on the whole boundary drive the system to rest in finite time. ***To cite this article: I. Lasiecka, B. Miara, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Contrôlabilité exacte d'un corps piézo-électrique. On considère un corps constitué d'un matériau dont la loi constitutive introduit un couplage élastique-électrique. On montre que, sans faire aucune hypothèse géométrique, l'application de deux contrôles agissant sur la totalité de la frontière (un contrôle élastique et un contrôle électrique) permet de contrôler le système en temps fini. ***Pour citer cet article : I. Lasiecka, B. Miara, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Version française abrégée

L'évolution d'un corps piézoélectrique donnée par un champ élastique $\boldsymbol{u}(x,t) = (u_i(x,t))$ et un potentiel électrique $\varphi(x,t)$, $x \in \bar{\Omega} \subset \mathbb{R}^3$, $0 \leqslant t \leqslant T$ se traduit par le problème hyperbolique-elliptique (1)–(2) où le tenseur des contraintes piezo-électrique $\mathcal{T}$ et le tenseur de déplacement électrique $\mathcal{D}$ sont reliés par la loi de comportement (4). Le but de cette Note est d'établir le résultat suivant :

Théorème 1. *Pour toute condition initiale $\boldsymbol{u}(0) \in \boldsymbol{L}_2(\Omega)$, $\boldsymbol{u}_t(0) \in \boldsymbol{H}^{-1}(\Omega)$ le système* (4) *peut être contrôlé exactement en un temps fini T avec deux contrôles $(\bar{\boldsymbol{u}}, \bar{\varphi})$ appliqués sur la frontière $\S = \partial\Omega \times (0,T)$, $\bar{\boldsymbol{u}} \in \boldsymbol{L}_2(\S)$, $\bar{\varphi} \in L_2(\S)$.*

Ce résultat permet de s'abstraire de l'hypothèse géométrique d'un domaine étoilé qui existait dans [7,9]. La difficulté essentielle réside dans le caractère elliptique de l'équation électrique qui conduit à un opérateur non local [2]. La

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démonstration consiste à remplacer le problème initial par le problème (6) en $\boldsymbol{u}$ seul, qu'on ré-écrit sous la forme (7) et dont la solution explicite [1,5,3] est donnée par (8). On fait ensuite appel à la théorie des semi-groupes [11,4] pour énoncer l'équivalence entre l'exacte contrôlabilité et l'inégalité (9). Cette dernière est établie grâce au Théorème 2. *L'inégalité* (9) *est équivalente à l'existence d'une constante positive* C_T *telle que* (11) *est vérifié pour toute solution* $\boldsymbol{v}$ *du problème homogène* (12).

1. Introduction: The evolution problem

Piezo-electric materials present, in their constitutive law, a coupling between the elastic strain and the electric gradient. However, contrary to the coupling in the thermo-elastic case in which the evolution equations introduce two evolution terms, viz., a second order evolution of the elastic displacement and a first order evolution of the temperature, in the case of piezo-electric materials only a second order evolution of the elastic displacement appears due to the assumption of instant propagation of the electric field. Let us describe precisely our framework. Let Ω be a domain (connected, bounded open subset) of $\mathbb{R}^3$ with a smooth boundary Γ. For $T > 0$, we denote by Q the domain $\Omega \times (0, T)$ and by § its boundary, $\S = \Gamma \times (0, T)$. The evolution problem in the elastic displacement[1] $\boldsymbol{u}(x,t) = (u_i(x,t))$ and the electric potential $\varphi(x,t)$, $x \in \bar{\Omega}$, $0 \leqslant t \leqslant T$ with Dirichlet boundary control $(\bar{\boldsymbol{u}}, \bar{\varphi})$ reads

$$\begin{cases} \rho \boldsymbol{u}_{tt} = \mathbf{div}\, \boldsymbol{\mathcal{T}}(\boldsymbol{u}, \varphi) & \text{in } Q, \\ -\mathbf{div}\, \boldsymbol{\mathcal{D}}(\boldsymbol{u}, \varphi) = 0 & \text{in } Q, \end{cases} \tag{1}$$

with Cauchy initial and Dirichlet boundary conditions

$$\begin{cases} \boldsymbol{u} = \bar{\boldsymbol{u}}, \quad \varphi = \bar{\varphi} & \text{on } \S, \\ \boldsymbol{u}(0) = \boldsymbol{u}^0, \quad \boldsymbol{u}'(0) = \boldsymbol{u}^1 & \text{in } \Omega, \end{cases} \tag{2}$$

where $\boldsymbol{\mathcal{T}}$ is the stress tensor, $\boldsymbol{\mathcal{D}}$ is the electric displacement, and the mass density ρ will be set, without any loss of generality, equal to 1 in the sequel. The constitutive law reads:

$$\begin{cases} \mathcal{T}^{ij}(\boldsymbol{u}, \varphi) = c^{ijkl} s_{kl}(\boldsymbol{u}) + e^{kij} \partial_k \varphi & \text{in } \Omega, \\ \mathcal{D}^i(\boldsymbol{u}, \varphi) = -e^{ikl} s_{kl}(\boldsymbol{u}) + d^{ij} \partial_j \varphi & \text{in } \Omega, \end{cases} \tag{3}$$

with the linearized deformation tensor $s_{kl}(\boldsymbol{u}) = \frac{1}{2}(\partial_l u_k + \partial_l u_k)$.

The controllability problem can be set as: Find a time $T > 0$ and a boundary control $(\bar{\boldsymbol{u}}, \bar{\varphi})$ to drive the system to rest at time T, i.e., $\boldsymbol{u}(t) = \boldsymbol{0}$, $\varphi(t) = 0$ for all $t \geqslant T$. The result presented in this Note, based on a direct approach, helps improve the previous ones [7,9] obtained by the Hilbert Uniqueness Method [6] with assumption of a star-shaped domain Ω; no such geometrical restriction is needed here. The direct approach consists first in re-writing the problem in terms of the electric displacement $\boldsymbol{u}$ only, and then in using semi-group theory. Other situations as controllability of shells [8], and 3D stabilization [10] for piezo-electric structures have also been considered.

In order to reformulate the evolution problem, we introduce the tensors $\boldsymbol{C} = (c^{ijkl})$, $\boldsymbol{E} = (e^{kij})$, $\boldsymbol{E}^* = (e^{ikl})$, $\boldsymbol{D} = (d^{ij})$, defined as follows:

$$\begin{cases} \boldsymbol{C}s(\boldsymbol{u}) = (\boldsymbol{C}s(\boldsymbol{u}))^{ij} = (c^{ijkl} s_{kl}(\boldsymbol{u})), \\ \boldsymbol{E}(\nabla\varphi) = (\boldsymbol{E}(\nabla\varphi))^{ij} = (e^{kij} \partial_k \varphi), \\ \boldsymbol{E}^* s(\boldsymbol{u}) = (\boldsymbol{E}^* s(\boldsymbol{u}))^i = (e^{ikl} s_{kl}(\boldsymbol{u})), \\ \boldsymbol{D}(\nabla\varphi) = (\boldsymbol{D}(\nabla\varphi))^i = (d^{ij} \partial_j \varphi). \end{cases}$$

The fourth order elasticity tensor (c^{ijkl}) is symmetric, positive definite and coercive, i.e., $c^{ijkl} = c^{jikl} = c^{klij}$ and there exists a positive constant α_c such that

$$c^{ijkl} X_{ij} X_{kl} \geqslant \alpha_c X_{ij} X_{ij}, \quad \forall X_{ij} = X_{ji} \in \mathbb{R}.$$

The third order coupling tensor (e^{ijk}) satisfies $e^{ijk} = e^{ikj}$.

[1] Latin exponents and indices take their values in the set $\{1, 2, 3\}$, Einstein convention for repeated exponents and indices is used and bold face letters represent vectors or vector spaces. Elastic displacements $(\boldsymbol{u}, \boldsymbol{v}, \ldots)$ will be represented with Latin letters, electric potentials $(\varphi, \psi, \ldots)$ with Greek letters and boundary terms will always be over lined. We denote the partial derivative with respect to time by subscript t and the divergence with respect to space variables by **div**; finally $\partial_l = \frac{\partial}{\partial x_l}$.

The second order dielectric tensor (d^{ij}) is symmetric, positive definite and coercive, i.e., $d^{ij} = d^{ji}$ and there exists a positive constant α_d such that

$$d^{ij} X_i X_j \geqslant \alpha_d X_i X_i, \quad \forall X_i \in \mathbb{R}.$$

In the sequel the coefficients of the three tensors are assumed to be constant.

With the previous notations the evolution problem reads

$$\begin{cases} \boldsymbol{u}_{tt} = \mathbf{div}\, \boldsymbol{C} s(\boldsymbol{u}) + \mathbf{div}\, \boldsymbol{E}(\nabla\varphi) & \text{in } Q, \\ \mathbf{div}\, \boldsymbol{D}(\nabla\varphi) = \mathbf{div}\, \boldsymbol{E}^* s(\boldsymbol{u}) & \text{in } Q \end{cases} \tag{4}$$

with the same boundary conditions (2). Our main result reads:

Theorem 1. *There exists $T > 0$ such that for any initial conditions $\boldsymbol{u}(0) \in \boldsymbol{L}_2(\Omega)$, $\boldsymbol{u}_t(0) \in \boldsymbol{H}^{-1}(\Omega)$ the evolution described by* (4) *can be controlled to rest in time $t > T$ with boundary controls $\bar{\boldsymbol{u}} \in \boldsymbol{L}_2(\Sigma)$, $\bar{\varphi} \in L_2(\Sigma)$.*

Remark 1. This result has been shown to hold in [7] for star-shaped domains. Theorem 1 removes this geometric restriction.

Remark 2. The difficulty associated with the study of controllability of evolutions such as in (4) is due to the hyperbolic-elliptic coupling in the model. The elliptic character of the electric equation leads to non-local operators, which typically have poor controllability properties [2].

2. Proof of Theorem 1

In order to study controllability problem we shall employ semigroup framework and convert the PDE system into a second order in time equation with non-local generators. To this end, we define several operators.

• Let us introduce the operator $A_{\boldsymbol{D}} : \eta \to A_{\boldsymbol{D}}\eta = \mathbf{div}\, \boldsymbol{D}(\nabla\eta)$ acting on $\mathrm{Dom}(A_{\boldsymbol{D}}) \subset L_2(\Omega) \to L_2(\Omega)$ with $\mathrm{Dom}(A_{\boldsymbol{D}}) = H^2(\Omega) \cap H^1_0(\Omega)$ and the corresponding "Dirichlet map" $G_{\boldsymbol{D}} : L_2(\Gamma) \to L_2(\Omega)$ defined as an "harmonic" extension of the boundary data $\bar{\eta} \in L_2(\Gamma)$ via the formula $\theta \equiv G_{\boldsymbol{D}}\bar{\eta}$ if and only if

$$\begin{cases} \mathbf{div}\, \boldsymbol{D}(\nabla\theta) = 0 & \text{in } \Omega, \\ \theta = \bar{\eta} & \text{on } \Gamma. \end{cases}$$

Standard elliptic theory gives $G_{\boldsymbol{D}} : H^s(\Gamma) \to H^{s+1/2}(\Omega)$ for every real s. Hence, the second equilibrium equation reads

$$\mathbf{div}\, \boldsymbol{D}(\nabla\varphi) = \mathbf{div}\, \boldsymbol{E}^* s(\boldsymbol{u}) = \mathbf{div}\, \boldsymbol{E}^* s(\boldsymbol{u}) + \mathbf{div}\, \boldsymbol{D}\nabla(G_{\boldsymbol{D}}\bar{\varphi}) \quad \text{in } Q.$$

Noting that $\varphi - G_{\boldsymbol{D}}\bar{\varphi} \in \mathrm{Dom}(A_{\boldsymbol{D}})$,

$$\mathbf{div}\, \boldsymbol{D}\nabla(\varphi - G_{\boldsymbol{D}}\bar{\varphi}) = A_{\boldsymbol{D}}(\varphi - G_{\boldsymbol{D}}\bar{\varphi})$$

we obtain $A_{\boldsymbol{D}}\varphi = \mathbf{div}\, \boldsymbol{E}^* s(\boldsymbol{u}) + A_{\boldsymbol{D}}(G_{\boldsymbol{D}}\bar{\varphi})$, defined on $[\mathrm{Dom}(A_{\boldsymbol{D}})]'$, and the electric field is explicitly given by

$$\varphi(\boldsymbol{u}, \bar{\varphi}) = A_{\boldsymbol{D}}^{-1}\, \mathbf{div}\, \boldsymbol{E}^* s(\boldsymbol{u}) + G_{\boldsymbol{D}}\bar{\varphi},$$

which, in turn, implies the compact form of the evolution equation:

$$\boldsymbol{u}_{tt} = \mathbf{div}\, \boldsymbol{C} s(\boldsymbol{u}) + \mathbf{div}\, \boldsymbol{E}\nabla\big(A_{\boldsymbol{D}}^{-1}\, \mathbf{div}\, \boldsymbol{E}^* s(\boldsymbol{u})\big) + \mathbf{div}\, \boldsymbol{E}\nabla G_{\boldsymbol{D}}\bar{\varphi}.$$

• In a similar manner we introduce the elliptic generator $\boldsymbol{A_E}$ associated with a non-local differential operator:

$$\boldsymbol{A_E}\boldsymbol{v} = \mathbf{div}\, \boldsymbol{C} s(\boldsymbol{v}) + \mathbf{div}\, \boldsymbol{E}\nabla\big(A_{\boldsymbol{D}}^{-1}\, \mathbf{div}\, \boldsymbol{E}^* s(\boldsymbol{v})\big),$$

$\boldsymbol{A_E}$ acts on $\mathrm{Dom}(\boldsymbol{A_E}) \subset \boldsymbol{L}_2(\Omega) \to \boldsymbol{L}_2(\Omega)$, its domain is $\mathrm{Dom}(\boldsymbol{A_E}) = \boldsymbol{H}^2(\Omega) \cap \boldsymbol{H}^1_0(\Omega)$, it is self adjoint and positive definite on $\boldsymbol{L}_2(\Omega)$ $(\mathrm{Dom}(\boldsymbol{A}_{\boldsymbol{E}}^{1/2}) = \boldsymbol{H}^1_0(\Omega))$, with the norm property[2]

$$\alpha|\boldsymbol{v}|_{\boldsymbol{H}^1(\Omega)} \leqslant \big|\boldsymbol{A}_{\boldsymbol{E}}^{1/2}\boldsymbol{v}\big| \leqslant \beta|\boldsymbol{v}|_{\boldsymbol{H}^1(\Omega)} \quad \forall \boldsymbol{v} \in \boldsymbol{H}^1_0(\Omega). \tag{5}$$

[2] $|\cdot|$ and $(\cdot,\cdot)$ represent respectively the norm and the scalar product in $L^2(\Omega)$.

As before, we introduce the Green's map $\boldsymbol{G_E}$ associated with the pseudo-differential action of $\boldsymbol{A_E}$, i.e., $\boldsymbol{G_E} : \bar{\boldsymbol{v}} \in \boldsymbol{L}_2(\Gamma) \to \boldsymbol{p} = \boldsymbol{G_E}\bar{\boldsymbol{v}} \in \boldsymbol{L}_2(\Omega)$ if and only if

$$\begin{cases} \mathbf{div}(\boldsymbol{C}s(\boldsymbol{p}) + \boldsymbol{E}\nabla A_{\boldsymbol{D}}^{-1}\,\mathbf{div}\,\boldsymbol{E}^* s(\boldsymbol{p})) = \boldsymbol{0} & \text{in } \Omega, \\ \boldsymbol{p} = \bar{\boldsymbol{v}} & \text{on } \Gamma. \end{cases}$$

Elliptic theory gives $\boldsymbol{G_E} : \boldsymbol{H}^{1/2}(\Gamma) \to \boldsymbol{H}^1(\Omega)$.

Now we can write the evolution problem as a non-local hyperbolic partial differential equation in $\boldsymbol{u}$ only:

$$\begin{cases} \boldsymbol{u}_{tt} = \boldsymbol{A_E}\boldsymbol{u} - \boldsymbol{A_E}(\boldsymbol{G_E}\bar{\boldsymbol{u}}) + \mathbf{div}\,\boldsymbol{E}\nabla(G_{\boldsymbol{D}}\bar{\varphi}) & \text{in } [\mathrm{Dom}(\boldsymbol{A_E})]', \\ \boldsymbol{u}(0) \in \boldsymbol{L}_2(\Omega), \quad \boldsymbol{u}_t(0) \in \boldsymbol{H}^{-1}(\Omega). \end{cases} \tag{6}$$

2.1. Semi-group theory and exact controllability

Associated with the boundary condition $\overline{\mathcal{U}}(t) = (\bar{\boldsymbol{u}}(t), \bar{\varphi}(t))$, we introduce the boundary operator

$$\boldsymbol{B} : \overline{\mathcal{U}} \longrightarrow \boldsymbol{B}\overline{\mathcal{U}} = -\boldsymbol{A_E}(\boldsymbol{G_E}\bar{\boldsymbol{u}}) + \mathbf{div}\,\boldsymbol{E}\nabla(G_{\boldsymbol{D}}\bar{\varphi}).$$

The control operator $\boldsymbol{B}$ is unbounded and takes $\overline{\mathcal{U}}$ into the dual to $\mathrm{Dom}\,A_{\boldsymbol{E}}$. With the notation $\mathcal{U}(t) = (\boldsymbol{u}(t), \boldsymbol{u}_t(t))$, $\mathcal{U}(0) = (\boldsymbol{u}(0), \boldsymbol{u}_t(0))$, the evolution problem (6) reads

$$\begin{cases} \mathcal{U}_t = \mathcal{A}\mathcal{U} + \mathcal{B}\overline{\mathcal{U}} & \text{in } Q, \\ \mathcal{U}(0) & \text{in } \Omega, \end{cases} \tag{7}$$

with $\mathcal{A} = (\boldsymbol{0}, \boldsymbol{I}/\boldsymbol{A_E}, \boldsymbol{0})$, $\mathcal{B}\overline{\mathcal{U}} = (\boldsymbol{0}, \boldsymbol{B}\overline{\mathcal{U}})$. The operator $\mathcal{A}$ with domain $\boldsymbol{L}_2(\Omega) \times [\mathrm{Dom}(\boldsymbol{A}_{\boldsymbol{E}}^{1/2})]'$ is skew adjoint, it generates a strongly continuous semigroup of contractions and the solution $\mathcal{U}(t)$ has the explicit form:

$$\mathcal{U}(t) = \mathrm{e}^{\mathcal{A}t}\mathcal{U}(0) + \int_0^t \mathrm{e}^{\mathcal{A}(t-r)}\mathcal{B}\overline{\mathcal{U}}(r)\,\mathrm{d}r \quad \text{in } \big[\mathrm{Dom}(\mathcal{A})\big]' \tag{8}$$

which is justified by well-known "boundary control variation of parameters formula" [1,5,3].

The exact controllability of (7) is equivalent [11,4] to the ontoness of the operator

$$\begin{cases} \boldsymbol{L}_T : \overline{\mathcal{V}}(t) = (\bar{\boldsymbol{v}}(t), \overline{\psi}(t)) \in \boldsymbol{L}_2(\S) \times L_2(\S) \longrightarrow \boldsymbol{L}_2(\Omega) \times \boldsymbol{H}^{-1}(\Omega), \\ \boldsymbol{L}_T\overline{\mathcal{V}} = \int_0^T \mathrm{e}^{\mathcal{A}(T-t)}\mathcal{B}\overline{\mathcal{V}}(t)\,\mathrm{d}t, \end{cases}$$

where, on the basis of formula (5), we consider an equivalent norm on $\boldsymbol{H}^{-1}(\Omega)$ by taking $|z|_{\boldsymbol{H}^{-1}(\Omega)} \sim |\boldsymbol{A}_{\boldsymbol{E}}^{-1/2}z|$. Exact controllability, by the closed range theorem [4], is equivalent to proving the following inequality,[3] stated for all $\mathcal{Z} = (z_1, z) \in \boldsymbol{L}_2(\Omega) \times \boldsymbol{H}^{-1}(\Omega)$,

$$\big|\boldsymbol{L}_T^*\mathcal{Z}\big|_{\boldsymbol{L}_2(\S)\times L_2(\S)} \geqslant C_T|\mathcal{Z}|_{\boldsymbol{L}_2(\Omega)\times[\mathrm{Dom}(\boldsymbol{A}_{\boldsymbol{E}}^{1/2})]'} = C_T\big(|z_1|^2 + |\boldsymbol{A}_{\boldsymbol{E}}^{-1/2}z|^2\big)^{\frac{1}{2}}. \tag{9}$$

In order to interpret this inequality in terms of a partial differential equation, let us consider the adjoint homogeneous backward problem with final conditions associated to $\mathcal{Z} = (z_1, z)$:

$$\begin{cases} \boldsymbol{y}_{tt}(t) = \boldsymbol{A_E}\boldsymbol{y}(t) & \text{in } Q, \\ \boldsymbol{y}(T) = \boldsymbol{A}_{\boldsymbol{E}}^{-1}z, \quad \boldsymbol{y}_t(T) = z_1 & \text{in } \Omega, \\ \bar{\boldsymbol{y}} = \boldsymbol{0} & \text{on } \S. \end{cases} \tag{10}$$

Then we establish the following result:

Lemma 1. *The controllability inequality* (9) *is equivalent to the existence of a positive constant C_T such that*

$$\big|\boldsymbol{G}_{\boldsymbol{E}}^*\boldsymbol{A_E}\boldsymbol{y}\big|_{\boldsymbol{L}_2(\Sigma)}^2 + \big|G_{\boldsymbol{D}}^*\nabla^*\boldsymbol{E}^*\,\mathbf{div}^*\boldsymbol{y}\big|_{L_2(\Sigma)}^2 \geqslant C_T\big(\big|\boldsymbol{A}_{\boldsymbol{E}}^{1/2}\boldsymbol{y}(T)\big|^2 + \big|\boldsymbol{y}_t(T)\big|^2\big)$$

holds for every solution $\boldsymbol{y}$ *to problem* (10) *associated to* $z_1 \in \boldsymbol{L}_2(\Omega), z) \in \boldsymbol{H}^{-1}(\Omega)$.

[3] From now on the superscript * represent the adjoint of an operator.

Proof. A straightforward computation yields the identification of the adjoint operator $\boldsymbol{L}_T^*$ as $\boldsymbol{L}_T^*\boldsymbol{\mathcal{Z}} = \boldsymbol{\mathcal{B}}^* \mathrm{e}^{-\boldsymbol{\mathcal{A}}(T-t)}\boldsymbol{\mathcal{Z}}$ for all $\boldsymbol{\mathcal{Z}} = (z_1, z) \in \boldsymbol{L}_2(\Omega) \times \boldsymbol{H}^{-1}(\Omega)$, and the computation of the adjoint $\boldsymbol{\mathcal{B}}^*$ yields

$$\begin{aligned}(\boldsymbol{\mathcal{B}}\overline{\boldsymbol{\mathcal{V}}}, \boldsymbol{\mathcal{Z}})_{\boldsymbol{L}_2(\Omega)\times[\mathrm{Dom}(\boldsymbol{A}_{\boldsymbol{E}}^{1/2})]'} &= (\boldsymbol{B}\overline{\boldsymbol{\mathcal{V}}}, z)_{[\mathrm{Dom}(\boldsymbol{A}_{BE}^{1/2})]'} = \big(\boldsymbol{A}_{\boldsymbol{E}}^{-1}\boldsymbol{B}\overline{\boldsymbol{\mathcal{V}}}, z\big), \\ &= -(\boldsymbol{G}_{\boldsymbol{E}}\bar{\boldsymbol{v}}, z) + \big(\boldsymbol{A}_{\boldsymbol{E}}^{-1}\,\mathbf{div}\,\boldsymbol{E}\nabla(G_{\boldsymbol{D}}\bar{\psi}), z\big).\end{aligned}$$

Therefore $\boldsymbol{\mathcal{B}}^*\boldsymbol{\mathcal{Z}} = (-\boldsymbol{G}_{\boldsymbol{E}}^* z, G_{\boldsymbol{D}}^*\nabla^*(\boldsymbol{E}^*\,\mathbf{div}^*\,\boldsymbol{A}_{\boldsymbol{E}}^{-1}z))$. We introduce the auxiliary function $\boldsymbol{\mathcal{W}}(t) = (\boldsymbol{w}_1, \boldsymbol{w}) = \mathrm{e}^{-\boldsymbol{\mathcal{A}}(T-t)}\boldsymbol{\mathcal{Z}}$, whose second component satisfies the second order backward evolution equation

$$\begin{cases}\boldsymbol{w}_{tt}(t) = \boldsymbol{A}_{\boldsymbol{E}}\boldsymbol{w}(t) & \text{in } Q,\\ \boldsymbol{w}(T) = z, \quad \boldsymbol{w}_t(T) = \boldsymbol{A}_{\boldsymbol{E}}z_1 & \text{in } \Omega,\\ \bar{\boldsymbol{w}} = \boldsymbol{0} & \text{on } \S.\end{cases}$$

Thus $(\boldsymbol{L}_T^*\boldsymbol{\mathcal{Z}})(t) = \boldsymbol{\mathcal{B}}^*\boldsymbol{\mathcal{W}}(t) = (-G_{\boldsymbol{E}}^*\boldsymbol{w}(t), G_{\boldsymbol{D}}^*\nabla^*(\boldsymbol{E}^*\,\mathbf{div}^*\,\boldsymbol{A}_{\boldsymbol{E}}^{-1}\boldsymbol{w}(t))$ and the lemma is obtained by letting $\boldsymbol{y}(t) = \boldsymbol{A}_{\boldsymbol{E}}^{-1}\boldsymbol{w}(t)$. □

2.2. Controllability inequality with two controls

We are now in a position to state the main technical result of this Note:

Theorem 2. *The controllability inequality* (9) *is equivalent to the existence of a positive constant C_T such that the following inequality*

$$\big|\boldsymbol{C}s(\boldsymbol{v})\cdot\boldsymbol{n} + \boldsymbol{E}\nabla\psi\cdot\boldsymbol{n}\big|_{\boldsymbol{L}^2(\S)} + |\boldsymbol{D}\nabla\psi\cdot\boldsymbol{n}|^2_{L^2(\S)} \geqslant C_T\big(\big|\boldsymbol{v}(0)\big|^2_{\boldsymbol{H}_0^1(\Omega)} + \big|\boldsymbol{v}_t(0)\big|^2\big) \tag{11}$$

holds for every solution $\boldsymbol{v}$ to the following problem associated to $z_1 \in \boldsymbol{L}_2(\Omega)$, $z \in \boldsymbol{H}^{-1}(\Omega)$:

$$\begin{cases}\boldsymbol{v}_{tt}(t) = \boldsymbol{A}_{\boldsymbol{E}}\boldsymbol{v}(t) & \text{in } Q,\\ \boldsymbol{v}(0) = \boldsymbol{A}_{\boldsymbol{E}}^{-1}z, \quad \boldsymbol{v}_t(0) = z_1 & \text{in } \Omega,\\ \bar{\boldsymbol{v}} = \boldsymbol{0} & \text{on } \S,\end{cases} \tag{12}$$

where ψ solves the elliptic problem $-\mathbf{div}\,\boldsymbol{\mathcal{D}}(\boldsymbol{v}, \psi) = 0$ in Q, $\psi = 0$ in S, and where $\boldsymbol{n}$ is the unit outer normal vector to Γ.

Proof. In (10) we consider the backward dynamics changing $\boldsymbol{y}(t) = \boldsymbol{v}(T-t)$. The controllability inequality in Lemma 1 now reads

$$\big|\boldsymbol{G}_{\boldsymbol{E}}^*\boldsymbol{A}_{\boldsymbol{E}}\boldsymbol{v}\big|^2_{\boldsymbol{L}^2(\S)} + \big|G_{\boldsymbol{D}}^*\nabla^*\boldsymbol{E}^*\,\mathbf{div}^*\boldsymbol{v}\big|^2_{L^2(\S)} \geqslant C_T\big(\big|\boldsymbol{A}_{\boldsymbol{E}}^{1/2}\boldsymbol{v}(0)\big|^2 + \big|\boldsymbol{v}_t(0)\big|^2\big).$$

Let us compute both terms of this sum, we get first

$$\begin{aligned}\big(\boldsymbol{G}_{\boldsymbol{E}}^*\boldsymbol{A}_{\boldsymbol{E}}\boldsymbol{v}, \bar{\boldsymbol{v}}\big)_{L^2(\S)} &= (\boldsymbol{A}_{\boldsymbol{E}}\boldsymbol{v}, \boldsymbol{G}_{\boldsymbol{E}}\bar{\boldsymbol{v}}) = \big(\mathbf{div}\big(\boldsymbol{C}s(\boldsymbol{v}) + \boldsymbol{E}\nabla A_{\boldsymbol{D}}^{-1}\,\mathbf{div}\,E^*s(\boldsymbol{v})\big), \boldsymbol{G}_{\boldsymbol{E}}\bar{\boldsymbol{v}}\big)\\ &= \big(\big(\boldsymbol{C}s(\boldsymbol{v}) + \boldsymbol{E}\nabla A_{\boldsymbol{D}}^{-1}\,\mathbf{div}\,\boldsymbol{E}^*s(\boldsymbol{v})\big)\cdot\boldsymbol{n}, \bar{\boldsymbol{v}}\big).\end{aligned}$$

Hence $\boldsymbol{G}_{\boldsymbol{E}}^*\boldsymbol{A}_{\boldsymbol{E}}\boldsymbol{v} = \boldsymbol{C}s(\boldsymbol{v})\cdot\boldsymbol{n} + \boldsymbol{E}\nabla\psi\cdot\boldsymbol{n}$ when reintroducing the electric potential ($\psi = A_{\boldsymbol{D}}^{-1}\,\mathbf{div}\,\boldsymbol{E}^*s(\boldsymbol{v})$). The second term reads

$$G_{\boldsymbol{D}}^*\nabla^*\boldsymbol{E}^*\,\mathbf{div}^*\boldsymbol{v} = G_{\boldsymbol{D}}^*A_{\boldsymbol{D}}\big(A_{\boldsymbol{D}}^{-1}\nabla^*\boldsymbol{E}^*\,\mathbf{div}^*\boldsymbol{v}\big) = G_{\boldsymbol{D}}^*A_{\boldsymbol{D}}\psi \quad \text{on } \S.$$

Finally by Green's formula, taking $\bar{\eta} \in L^2(\Gamma)$ with $\boldsymbol{A}_{\boldsymbol{D}}G_{\boldsymbol{D}}\bar{\eta} = 0$ in Ω and recalling that $\psi = 0$ on §, we get $(G_{\boldsymbol{D}}^*A_{\boldsymbol{D}}\psi, \bar{\eta}) = (\boldsymbol{D}\nabla\psi\cdot\boldsymbol{n}, \bar{\eta})$. The equivalence of the norms in $H_0^1(\Omega)$ and in $\mathrm{Dom}(\boldsymbol{A}_{\boldsymbol{E}}^{1/2})$ completes the argument. □

2.3. Completion of the proof of Theorem 1

Inequality in Theorem 1 above has been established in [7]. This, along with Theorem 2, completes the proof of Theorem 1.

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