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Differential Geometry

Remarks on surfaces of large mean curvature

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Abstract

Let M be a complete embedded H-surface of bounded curvature in N^3, a homogeneously regular 3-manifold. We prove that if H is large (in terms of the scalar curvature of N) then M is properly embedded. The proof follows from two theorems. First, if M is a complete stable immersed H-surface in N and H is large, then M is topologically a sphere. Secondly, a theorem of Meeks, Perez and Ros is used: limit leaves of *CMC*-laminations are stable. ***To cite this article: H. Rosenberg, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Remarques sur les surfaces de courbure moyenne grande. Soit M une surface complète de courbure moyenne constante H dans N^3, de courbure bornée. On montre que si H est grand (par rapport à la courbure scalaire de N) alors M est proprement plongée. La preuve utilise deux théorèmes. Le premier est que si M est est une surface de courbure moyenne constante stable dans N (complète) et si H est grand, alors M est topologiquement une sphère. Le second est un théorème de Meeks, Perez et Ros : Les feuilles limites d'une lamination *CMC* sont stables. ***Pour citer cet article : H. Rosenberg, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

1. Introduction

Let N be an orientable homogeneously regular 3-manifold. This mean there is some positive R so that the geodesic balls of N of radius R, centered at any point of N are embedded, and in these balls, the sectional curvatures are bounded by a constant independent of the point of N where the balls are centered. We will prove:

Theorem 1.1. *Let $c > 0$ and H be constants satisfying*

$$3H^2 + S(x) \geqslant c,$$

where S is the scalar curvature function of N. Then a complete embedded H-surface M in N, of bounded curvature, is properly embedded.

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Remark 1. A hypothesis on the size of H is necessary; consider a foliation of a flat 3-torus by dense planes ($H = 0$). Also notice that when N has positive Ricci curvature bounded away from zero (so N is compact) then the theorem implies a complete embedded H-surface in N of bounded curvature is compact.

The proof is an application of two theorems (more precisely, of one theorem and the proof of the second theorem). The author proved in [2], the following theorem:

Theorem 1.2. *Let N be an orientable homogeneously regular* 3*-manifold and let M be a complete orientable surface immersed in N of constant mean curvature H. Suppose M is a* (*strongly*) *stable H-surface and*

$$3H^2 + S(x) \geqslant c > 0.$$

Then M is topologically a sphere.

Remark 2. The above theorem is an immediate consequence of the proof of Theorem 1 of [2].

We now explain this. The exact statement of Theorem 1 of [2], is that for surfaces M as in Theorem 1.2, and allowing ∂M to be non-empty (until now, M complete meant empty boundary), then for $p \in M$,

$$\operatorname{dist}_M(p, \partial M) \leqslant \frac{2\pi}{\sqrt{3c}}.$$

In particular, when $\partial M = \emptyset$, then M is compact. The proof of Theorem 1 of [2] involves studying the metric $\mathrm{d}\tilde{s}^2 = u\,\mathrm{d}s^2$, where $\mathrm{d}s^2$ is the metric on M and u is a Jacobi function on M coming from stability, i.e., $u > 0$ is in the kernel of the linearized operator

$$L = \Delta + |A|^2 + \operatorname{Ric}(n),$$

where Δ is the Laplacian of the metric M, A the second fundamental form, and n a unit normal to M. It is shown in [2], that the metric $\mathrm{d}\tilde{s}^2$ has positive intrinsic curvature, so M is an immersed sphere when M is compact.

The second theorem we need for the proof of Theorem 1.1, says that a limit leaf of a *CMC* lamination, is (strongly) stable [1].

Now we can prove Theorem 1.1. If M is not properly embedded, then the closure of M is a *CMC* lamination with a limit leaf M_0, and M_0 is stable [1]. By Theorem 1.2, M_0 is an immersed sphere.

Now M spirals towards M_0 ($M_0 \subset \bar{M}$ and $M_0 \neq M$), so one can lift paths on M_0 into paths of M. Since M_0 is compact and simply connected, lifting paths defines an immersion of M_0 into M. The image of M_0 is open and closed in M so M is compact; a contradiction. This proves Theorem 1.1.

We remark that M_0 may not be embedded, there may be isolated points of M_0 where two sheets of M_0 touch on their mean concave sides. But this poses no obstacle to lifting paths of M_0 to M.

Remark 3. Suppose N is a compact 3-manifold with positive Ricci curvature. Then if M is a complete embedded H-surface in N, of bounded curvature, we have M is compact. Otherwise the lamination $\bar{M}$ would have a stable leaf M_0 by the theorem of Meeks, Perez and Ros. Then M_0 is a compact sphere by Theorem *1.2*. Stability means $-\int_{M_0} f L(f) \geqslant 0$, for all f of compact support on M_0. Take $f = 1$ on M_0, then

$$L(f) = \Delta f + \big(|A|^2 + \operatorname{Ric}\big)(n) > 0;$$

a contradiction.

References

[1] W. Meeks, J. Perez, A. Ros, Limit leaves of a *CMC* lamination are stable, arXiv: 0801.4345v1 [math.DG], 28 Jan 2008.
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