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Partial Differential Equations

Boundedness of the negative part of biharmonic Green's functions under Dirichlet boundary conditions in general domains

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Abstract

In general, higher order elliptic equations and boundary value problems like the biharmonic equation or the linear clamped plate boundary value problem do not enjoy neither a maximum principle nor a comparison principle or – equivalently – a positivity preserving property. It is shown that, on the other hand, for bounded smooth domains $\Omega \subset \mathbb{R}^n$, the negative part of the corresponding Green's function is "small" when compared with its singular positive part, provided that $n \geqslant 3$. ***To cite this article: H.-Ch. Grunau, F. Robert, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Borne L^∞ pour la partie négative de la fonction de Green biharmonique avec condition de Dirichlet au bord d'un domaine arbitraire. De manière générale, les équations elliptiques de grand ordre et les problèmes aux limites correspondant (comme l'équation biharmonique ou bien l'équation des plaques encastrées) ne satisfont ni un principe du maximum, ni un principe de comparaison ou bien, de façon équivalente, une propriété de conservation de la positivité. En revanche, nous montrons que pour des domaines bornés réguliers de $\mathbb{R}^n$, la partie négative de la fonction de Green correspondante est « petite » comparée à la partie positive singulière dès que $n \geqslant 3$. ***Pour citer cet article : H.-Ch. Grunau, F. Robert, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

1. Introduction

Although simple examples show that strong maximum principles as satisfied e.g. by harmonic functions cannot hold true for solutions of higher order elliptic equations, it is reasonable to ask whether higher order *boundary value problems* may possibly enjoy a *positivity preserving property*. To be specific, we consider the clamped plate boundary value problem:

$$\Delta^2 u = f \quad \text{in } \Omega, \qquad u_{|\partial\Omega} = \left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega} = 0. \tag{1}$$

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Here $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain with exterior unit normal ν at $\partial\Omega$, and f is a sufficiently smooth datum. We say that (1) enjoys a *positivity preserving property* in Ω, if $f \geqslant 0$ always implies that also the solution is nonnegative, i.e. $u \geqslant 0$. Equivalently, one may ask whether the corresponding Green's function $H_\Omega := H_{\Omega,\Delta^2}$ is nonnegative or even strictly positive.

Boggio [1] (1901) and Hadamard [14] (1908) conjectured that in arbitrary convex (two dimensional) domains Ω, the positivity preserving property should hold true. Boggio [2] could show with the help of a beautiful explicit formula that this is indeed the case for balls in $\mathbb{R}^n$. At the same time Hadamard already knew that sign change occurs in annuli with small inner radius.

Starting about 40 years later, numerous counterexamples disproved the Boggio–Hadamard conjecture, see e.g. Duffin [6] and Garabedian [9]. A more extensive survey and further references can be found in Grunau–Sweers [11]. Nevertheless, motivated by discussions with physicists and engineers, one may ask:

> Is positivity preserving in any bounded smooth domain possibly "almost true" in the sense that the negative part $H_\Omega^-(x,y) := \min\{H_\Omega(x,y), 0\}$ of the biharmonic Green's function under Dirichlet boundary conditions is "small relatively" to the singular positive part $H_\Omega^+(x,y) := \max\{H_\Omega(x,y), 0\}$?

In order to explain in which sense we are able to answer this question in the affirmative, let us consider any bounded sufficiently smooth domain $\Omega \subset \mathbb{R}^n$ $(n \geqslant 2)$ and the Green's function H_Ω in Ω for the clamped plate boundary value problem (1). For this general case, Dall'Acqua and Sweers [4] deduced from Krasovskiĭ's work [15] the following estimates, where $d(x) = \operatorname{dist}(x, \partial\Omega)$ and $C = C(\Omega) > 0$ denotes a constant:

$$\left|H_\Omega(x,y)\right| \leqslant \begin{cases} C \cdot |x-y|^{4-n} \min\left\{1, \dfrac{d(x)^2 d(y)^2}{|x-y|^4}\right\}, & \text{if } n > 4, \\ C \cdot \log\left(1 + \dfrac{d(x)^2 d(y)^2}{|x-y|^4}\right), & \text{if } n = 4, \\ C \cdot d(x)^{2-n/2} d(y)^{2-n/2} \min\left\{1, \dfrac{d(x)^{n/2} d(y)^{n/2}}{|x-y|^n}\right\}, & \text{if } n = 2, 3. \end{cases} \tag{2}$$

As far as the positive part H_Ω^+ is concerned, these estimates cannot be improved, see e.g. Grunau–Sweers [12]. However, they do not distinguish between the positive and the negative part of the Green's function. A distinction between H_Ω^+ and H_Ω^- of order $|x-y|^{-n}$ is the subject of our main result:

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ $(n \geqslant 3)$ be a bounded $C^{4,\alpha}$-smooth domain. We denote by H_Ω the biharmonic Green's function in Ω under Dirichlet boundary conditions.*

Then, there exists a constant $\delta = \delta(\Omega) > 0$ such that for any two points $x, y \in \Omega$, $x \neq y$,

$$H_\Omega(x,y) \leqslant 0 \text{ implies that } |x-y| \geqslant \delta. \tag{3}$$

In particular, if Ω is smooth enough for (2) *to hold true, there exists a constant $C = C(\Omega) > 0$ such that for all $x, y \in \Omega$, $x \neq y$, we have the estimate from below*:

$$H_\Omega(x,y) \geqslant -C d(x)^2 d(y)^2. \tag{4}$$

We are not able to extend our method of proving (3) to $n = 2$. Like Nehari in [16] we cannot exclude the possibility of a two dimensional domain, where one might find sequences $x_k, y_k \in \Omega$, $x_k, y_k \to x_\infty \in \partial\Omega$ with $H_\Omega(x_k, y_k) = 0$. We can even not exclude this situation in the case where $d(x_k)$, $d(y_k)$ and $|x_k - y_k|$ converge to 0 on the same scale.

The bound (4), however, was proved for the case $n = 2$ by Dall'Acqua, Meister and Sweers [3] by different methods taking advantage of conformal maps. Even for $n = 2, 3$, where the Green's function is bounded, (4) is a strong statement because in the case, where x or y is closer to the boundary than they are to each other, (2) would only give $H_\Omega \geqslant -C\frac{d(x)^2 d(y)^2}{|x-y|^n}$. In this sense, (4) gains a factor of order $|x-y|^n$.

Garabedian [9] proved change of sign for H_Ω with Ω a mildly eccentric ellipse by finding opposite boundary points $x_0, y_0 \in \partial\Omega$ with $\Delta_x \Delta_y H_\Omega(x_0, y_0) < 0$. This shows that qualitatively, the estimate (4) cannot be further improved.

Local positivity has been also studied in the Cauchy problem for the corresponding parabolic equation $u_t + \Delta^2 u = 0$, see [5] and also [8]. As the rescaling argument will explain, the whole space corresponds to the case in Theorem 1.1 where x and y are closer to each other than to the boundary $\partial\Omega$. The emphasis in our result, however, is laid on the reverse case. The main difficulty in proving (3) consists in gaining uniformity while the pole of the Green's function approaches the boundary.

2. The rescaling argument

Theorem 1.1 is embedded in [10] into a more general context, where also smooth perturbations of domains are studied. The key issue in the proofs is a rescaling analysis which we outline here in order to sketch the proof of Theorem 1.1 for the case $n > 4$. For details we refer to [10].

In order to prove (3) we assume by contradiction that there exist $x_k, \tilde{y}_k \in \Omega$, $x_k \neq \tilde{y}_k$ with $H_\Omega(x_k, \tilde{y}_k) \leqslant 0$ and $\lim_{k\to\infty} |x_k - \tilde{y}_k| = 0$. In view of the smoothness assumption on Ω there exist $y_k \in \Omega$, $x_k \neq y_k$ with $H_\Omega(x_k, y_k) = 0$ and $|x_k - y_k| \to 0$ for $k \to \infty$. After passing to a subsequence we find $x_\infty \in \overline{\Omega}$ with $x_\infty = \lim_{k\to\infty} x_k = \lim_{k\to\infty} y_k$.

In what follows we shall rescale so that in the limit, after passing to a further suitable subsequence, we shall obtain a space or a half space and deduce a contradiction to the properties of the corresponding biharmonic Green's function. A key issue in this limit procedure is the following estimate

$$\left|H_\Omega(x, y)\right| \leqslant C|x - y|^{4-n}, \tag{5}$$

for all $x, y \in \Omega$, $x \neq y$, where $C = C(\Omega)$. See Krasovskiĭ [15], cf. also [10, Theorem 4]. We consider the rescaled family of Green's functions

$$\widetilde{H}_k(z, \zeta) := |x_k - y_k|^{n-4} H_\Omega\big(x_k + |x_k - y_k|z, x_k + |x_k - y_k|\zeta\big) \quad \text{for } z, \zeta \in \Omega_k := \frac{1}{|x_k - y_k|}(-x_k + \Omega).$$

After possibly passing to a further subsequence one may prove (working in coordinate charts if necessary) that $\Omega_k \to \mathcal{H}$ in $C^{4,\alpha}_{\text{loc}}$, where

either $\mathcal{H} = \mathbb{R}^n$ or $\mathcal{H}$ is a half-space of $\mathbb{R}^n$.

In view of (5) one has uniformly in k that $|\widetilde{H}_k(z, \zeta)| \leqslant C|z - \zeta|^{4-n}$ for all $z, \zeta \in \Omega_k$. This estimate allows to prove by making use of biharmonic reflection principles (see Duffin [7]) and biharmonic Liouville theorems (see Nicolesco [17]) that $\widetilde{H}_k \to H_{\mathcal{H}}$ in a suitable sense, where $H_{\mathcal{H}}$ is the Green's function for Δ^2 with Dirichlet boundary condition and decaying at infinity on $\mathcal{H}$. For a detailed exposition of these arguments we refer to [10, Lemma 6.5]. Since

$$\widetilde{H}_k(0, \eta_k) = 0, \quad \text{where } \eta_k := -\frac{1}{|x_k - y_k|}(x_k - y_k),$$

after passing to a further subsequence, one finds $\eta \in \overline{\mathcal{H}}$ such that $|\eta| = 1$ and, by working in coordinate charts, that one of the following three cases occurs:

(i) $0 \in \mathcal{H}$, $\eta \in \mathcal{H}$, $H_{\mathcal{H}}(0, \eta) = 0$;
(ii) $0 \in \mathcal{H}$, $\eta \in \partial\mathcal{H}$, $\Delta_\zeta H_{\mathcal{H}}(0, \eta) = 0$, or vice versa;
(iii) $0 \in \partial\mathcal{H}$, $\eta \in \partial\mathcal{H}$, $\Delta_z \Delta_\zeta H_{\mathcal{H}}(0, \eta) = 0$.

All three cases are impossible: for instance if $\mathcal{H} = \mathbb{R}^n$, we have that

$$H_{\mathbb{R}^n}(z, \zeta) = c_n|z - \zeta|^{4-n} \quad \text{for all } z, \zeta \in \mathbb{R}^n,$$

where $c_n > 0$ is an explicit constant depending only on the dimension n. In this case a contradiction can be obtained also differently by using the local positivity result of Grunau–Sweers [13] (cf. Nehari [16] for $n = 3$). In case that $\mathcal{H}$ is a half-space we know that, in view of an explicit formula due to Boggio [2], the biharmonic Green's function $H_{\mathcal{H}}$ is strictly positive and does not display any degenerate behaviour on the boundary $\partial\mathcal{H}$. This contradiction concludes the proof of (3), which in view of (2) implies the estimate (4). This concludes the proof of Theorem 1.1 in the case $n > 4$. The case $n = 3, 4$ is technically more involved and requires to estimate also the derivatives of the Green's functions. See [10].

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