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C. R. Acad. Sci. Paris, Ser. I 347 (2009) 255–260

COMPTES RENDUS MATHEMATIQUE

Partial Differential Equations

$W^{1,N}$ versus C^1 local minimizers for elliptic functionals with critical growth in $\mathbb{R}^N$

Jacques Giacomoni [a], S. Prashanth [b], K. Sreenadh [c]

[a] *Université de Pau et des Pays de l'Adour, B.P. 576, 64012 Pau cedex, France*
[b] *T.I.F.R. CAM, P.B. No. 6503, Sharadanagar, Chikkabommasandra, Bangalore 560065, India*
[c] *Department of Mathematics, Indian Institute of Technology Delhi, Hauz Khaz, New Delhi-16, India*

Received 2 September 2008; accepted after revision 12 January 2009

Available online 25 February 2009

Presented by Haïm Brezis

Abstract

Let $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Caratheodory function with $sf(x,s) \geqslant 0\ \forall (x,s) \in \Omega \times \mathbb{R}$ and $\sup_{x\in\Omega} |f(x,s)| \leqslant C(1+|s|)^p \mathrm{e}^{|s|^{N/(N-1)}}$, $\forall s \in \mathbb{R}$, for some $C > 0$. Consider the functional $J : W^{1,N}(\Omega) \to \mathbb{R}$, Ω defined as

$$J(u) \stackrel{\text{def}}{=} \frac{1}{N}\|u\|^N_{W^{1,N}(\Omega)} - \int_\Omega F(x,u) - \frac{1}{q+1}\|u\|^{q+1}_{L^{q+1}(\partial\Omega)},$$

where $F(x,u) = \int_0^u f(x,s)\,\mathrm{d}s$ and $q > 0$. We show that if $u_0 \in C^1(\overline{\Omega})$ is a local minimum of J in the $C^1(\overline{\Omega})$ topology, then it is also a local minimum of J in $W^{1,N}(\Omega)$ topology. ***To cite this article: J. Giacomoni et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Minima locaux relatifs à C^1 et à $W^{1,N}$. Soit Ω un ouvert borné régulier de $\mathbb{R}^N$, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ une fonction de Caratheodory vérifiant $sf(x,s) \geqslant 0\ \forall (x,s) \in \Omega \times \mathbb{R}$ et $\sup_{x\in\Omega} |f(x,s)| \leqslant C(1+|s|^p)\mathrm{e}^{|s|^{N/(N-1)}}$, $\forall s \in \mathbb{R}$ et pour une constante $C > 0$. Considérons la fonctionnelle $J : W^{1,N}(\Omega) \to \mathbb{R}$, définie par

$$J(u) \stackrel{\text{def}}{=} \frac{1}{N}\|u\|^N_{W^{1,N}(\Omega)} - \int_\Omega F(x,u) - \frac{1}{q+1}\|u\|^{q+1}_{L^{q+1}(\partial\Omega)}$$

avec $F(x,u) = \int_0^u sf(x,s)\,\mathrm{d}s$ et $q > 0$. Nous démontrons que si $u_0 \in C^1(\overline{\Omega})$ est un minimiseur local de J dans $C^1(\overline{\Omega})$, alors il est aussi un minimiseur local de J dans $W^{1,N}(\Omega)$. ***Pour citer cet article : J. Giacomoni et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

E-mail addresses: jgiacomo@univ-pau.fr (J. Giacomoni), pras@math.tifrbng.res.in (S. Prashanth), sreenadh@maths.iitd.ernet.in (K. Sreenadh).

1631-073X/$ – see front matter
doi:10.1016/j.crma.2009.01.010

Version française abrégée

Soit $\Omega \subset \mathbb{R}^N$, avec $N \geqslant 2$, un ouvert borné régulier. Soit $f : \Omega \times \mathbb{R} \to \mathbb{R}$ une fonction de Caratheodory satisfaisant :

(f1) Il existe $p > 1$ tel que $|f(x,s)| \leqslant C(1+|s|)^p e^{|s|^{N/(N-1)}}$ pour tout $(x,s) \in \Omega \times \mathbb{R}$ et pour une certaine constante $C > 0$,
(f2) $sf(x,s) \geqslant 0$ pour tout $(x,s) \in \Omega \times \mathbb{R}$.

Soit $F(x,u) \stackrel{\text{def}}{=} \int_0^u f(x,s)\,ds$ et $q > 0$. On considère la fonctionnelle $J : W^{1,N}(\Omega) \to \mathbb{R}$ définie par (1). L'objectif de la présente Note est de démontrer le théorème suivant :

Théorème 0.1. *Soit $u_0 \in C^1(\overline{\Omega})$ un minimiseur local de J dans la topologie $C^1(\overline{\Omega})$, ce qui signifie que*

$$\exists \delta > 0 \quad \text{tel que} \quad \|u - u_0\|_{C^1(\overline{\Omega})} < \delta \Rightarrow J(u_0) \leqslant J(u).$$

Alors u_0 est aussi un minimiseur local de J dans la topologie $W^{1,N}(\Omega)$.

Pour prouver le résultat précédent, nous avons besoin d'estimations uniformes dans L^∞ pour une famille de solutions du problème (P_ϵ) (défini dans la section suivante). Précisément, nous utilisons le résultat suivant que nous démontrons dans la section 3 :

Théorème 0.2. *Soit $\{u_\epsilon\}_{\epsilon \in (0,1)}$ une famille de solutions de (P_ϵ) et u_0 une solution de* (3). *Soit $\theta > 1$ tel que* $\sup_{\epsilon \in (0,1)}(\|f(x,u_\epsilon)\|_{L^\theta(\Omega)} + \|u_\epsilon\|_{W^{1,N}(\Omega)}) < \infty$. *Alors,* $\sup_{\epsilon \in (0,1)} \|u_\epsilon\|_{L^\infty(\Omega)} < \infty$.

Un ingrédient important dans la preuve de ce résultat est l'inégalité de Trudinger–Moser rappelée en (4).

1. Introduction and main results

Let $\Omega \subset \mathbb{R}^N$, $N \geqslant 2$ be a bounded smooth domain. Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Caratheodory function satisfying:

(f1) There exists $p > 1$ such that $|f(x,s)| \leqslant C(1+|s|)^p e^{|s|^{N/(N-1)}}$ for all $(x,s) \in \Omega \times \mathbb{R}$ for some $C > 0$,
(f2) $sf(x,s) \geqslant 0$ for all $(x,s) \in \Omega \times \mathbb{R}$.

Let $F(x,u) \stackrel{\text{def}}{=} \int_0^u f(x,s)\,ds$ and $q > 0$. We consider the functional $J : W^{1,N}(\Omega) \to \mathbb{R}$ given by

$$J(u) \stackrel{\text{def}}{=} \frac{1}{N}\|u\|^N_{W^{1,N}(\Omega)} - \int_\Omega F(x,u) - \frac{1}{q+1}\int_{\partial\Omega} |u|^{q+1}. \tag{1}$$

Our aim in this Note is to show the following:

Theorem 1.1. *Let $u_0 \in C^1(\overline{\Omega})$ be a local minimizer of J in $C^1(\overline{\Omega})$ topology. That is,*

$$\exists \delta > 0 \quad \text{such that} \quad \|u - u_0\|_{C^1(\overline{\Omega})} < \delta \Rightarrow J(u_0) \leqslant J(u). \tag{2}$$

Then u_0 is a local minimum of J in $W^{1,N}(\Omega)$ topology also.

We remark here that the above theorem is valid when J is restricted to the subspace $W_0^{1,N}(\Omega)$. That is, any C^1 local minimiser of such a J will be a local minimiser in $W_0^{1,N}(\Omega)$. The proof of this case is very similar, in fact simpler, to the proof given below. Also, essentially the same proof goes through when we replace the $|u|^{q+1}$ term in J by a more general boundary term $h(x,u)$ that has similar asymptotic behaviour.

Let $u_0 \in C^1(\overline{\Omega})$ solve

$$\begin{cases} -\Delta_N u_0 + |u_0|^{N-2}u_0 = f(x,u_0) & \text{in } \Omega, \\ |\nabla u_0|^{N-2}\dfrac{\partial u_0}{\partial \nu} = |u_0|^{q-1}u_0 & \text{on } \partial\Omega. \end{cases} \tag{3}$$

For proving the above theorem, we will need uniform L^∞-estimates for a family of solutions to (P_ϵ) (see Section 2) as below.

Theorem 1.2. *Let $\{u_\epsilon\}_{\epsilon\in(0,1)}$ be a family of solutions to (P_ϵ), where u_0 solves* (3). *Let $\theta > 1$ be such that $\sup_{\epsilon\in(0,1)}(\|f(x,u_\epsilon)\|_{L^\theta(\Omega)} + \|u_\epsilon\|_{W^{1,N}(\Omega)}) < \infty$. Then, $\sup_{\epsilon\in(0,1)}\|u_\epsilon\|_{L^\infty(\Omega)} < \infty$.*

An important ingredient in our proof is the following Trudinger–Moser type inequality (see [5,6]):

$$\sup\left\{\alpha \;\middle|\; \sup_{\|u\|_{W^{1,N}(\Omega)}\leqslant 1}\int_\Omega \mathrm{e}^{\alpha|u|^{N/(N-1)}} < \infty\right\} = \frac{N}{2}w_{N-1}^{1/(N-1)}, \quad w_{N-1} = \mathrm{Volume}\big(S^{N-1}\big). \tag{4}$$

Theorem 1.1 was proved first in [2] for the case of critical growth functionals $J : H_0^1(\Omega) \to \mathbb{R}$, $\Omega \subset \mathbb{R}^N$, $N \geqslant 3$, and later for critical growth functionals $J : W_0^{1,p}(\Omega) \to \mathbb{R}$, $1 < p < N$, $\Omega \subset \mathbb{R}^N$, $N \geqslant 3$ in [3]. A key feature of the latter work is the uniform $C^{1,\alpha}$ estimate they obtain for equations like (P_ϵ) but involving two p-Laplace operators. Using constraints based on L^p-norms rather than Sobolev norms as in [3], the equations for which uniform estimates required can be simplified to a standard type involving only one p-Laplace operator. This approach was followed in [4] which is also adopted in this work. We remark that such "Sobolev versus C^1 local minimizers" results find application in proving existence of at least two positive solutions to "concave-convex" type problems (see [1–4,7,8]). Indeed, in a forthcoming article, we use Theorem 1.1 to prove multiplicity of positive solutions to critical growth problems with co-normal boundary conditions.

2. Sobolev versus C^1 local minimizers

Proof of Theorem 1.1. Clearly, u_0 is a local minimum of J in $C^1(\overline{\Omega})$ (resp. $W^{1,N}(\Omega)$) if and only if 0 is a local minimizer of the functional $J(\cdot + u_0)$ in $C^1(\overline{\Omega})$ (resp. $W^{1,N}(\Omega)$). Hence it is enough to show, assuming that 0 is a local minimizer of $J(\cdot + u_0)$ in $C^1(\overline{\Omega})$ that 0 is also a local minimizer of $J(\cdot + u_0)$ in $W^{1,N}(\Omega)$. We prove this statement by a contradiction argument. Suppose 0 is not a local minimizer of $J(\cdot + u_0)$ in $W^{1,N}(\Omega)$. Then, there exists a sequence $\{v_n\}_{n\geqslant 1} \subset W^{1,N}(\Omega)$ such that

$$\|v_n\|_{W^{1,N}(\Omega)} \leqslant \frac{1}{n} \quad \text{and} \quad J(u_0 + v_n) < J(u_0) \quad \forall n \geqslant 1. \tag{5}$$

Let $G : \mathbb{R} \to \mathbb{R}$ be defined as $G(s) = |s|^{p+1}\mathrm{e}^{2s^{N/(N-1)}}$. We define the following constraint for each $\epsilon > 0$:

$$\mathcal{C}_\epsilon \stackrel{\text{def}}{=} \big\{u \in W^{1,N}(\Omega) : K(u) \stackrel{\text{def}}{=} \big\|G(u)\big\|_{L^1(\Omega)} + \|u\|^{\alpha+1}_{L^{\alpha+1}(\partial\Omega)} \leqslant \epsilon\big\}, \quad \alpha \stackrel{\text{def}}{=} \max\{p,q\}. \tag{6}$$

Therefore, $K(v_n) \to 0$ as $n \to \infty$, thanks to properties in (5) and the Moser–Trudinger embedding (4). This shows that for any $\epsilon \in (0,1)$, there exists $N_\epsilon \in \mathbb{N}$ such that $v_n \in \mathcal{C}_\epsilon$ for $n \geqslant N_\epsilon$. In particular, $\mathcal{C}_\epsilon \neq \emptyset$ $\forall\epsilon \in (0,1)$. Clearly, the following coercivity property of J holds on $\mathcal{C}_\epsilon$:

$$J(u+u_0) \geqslant \frac{1}{N}\int_\Omega \big|\nabla(u+u_0)\big|^N + |u+u_0|^N - C\epsilon, \quad u \in \mathcal{C}_\epsilon,\ \epsilon \in (0,1). \tag{7}$$

We note that $\mathcal{C}_\epsilon$ is a convex set. Using Trudinger–Moser and trace embeddings we see that $\mathcal{C}_\epsilon$ is also a closed set in $W^{1,N}(\Omega)$ which implies that $\mathcal{C}_\epsilon$ is weakly closed in $W^{1,N}(\Omega)$. Therefore, by (5) and the fact $v_n \in \mathcal{C}_\epsilon$ for some n, we can find $u_\epsilon \in \mathcal{C}_\epsilon$ such that $u_\epsilon \not\equiv 0$ and

$$\min_{u\in\mathcal{C}_\epsilon} J(u+u_0) = J(u_\epsilon + u_0) < J(u_0), \quad \epsilon \in (0,1). \tag{8}$$

Clearly from (7) and (8) we get that $\{u_\epsilon\}_{\epsilon\in(0,1)}$ is a bounded sequence in $W^{1,N}(\Omega)$. Since $K(u_\epsilon) \leqslant \epsilon$, we get that as $\epsilon \to 0^+$, $u_\epsilon \to 0$ pointwise a.e. in Ω. Therefore, $u_\epsilon + u_0 \rightharpoonup u_0$ in $W^{1,N}(\Omega)$. From (8), using the Lagrange multiplier rule, we obtain that u_ϵ solves

$$J'(u_\epsilon + u_0) = \mu_\epsilon K'(u_\epsilon) \quad \text{for some } \mu_\epsilon \in \mathbb{R},\ \forall\epsilon \in (0,1). \tag{9}$$

We now claim that $\mu_\epsilon \leqslant 0\ \forall \epsilon \in (0,1)$. Suppose $\mu_\epsilon > 0$ for some $\epsilon > 0$. We choose $\phi \in W^{1,N}(\Omega)$ such that $J'(u_\epsilon + u_0)\phi < 0$ (possible since $K'(u_\epsilon) \neq 0$) which implies from (9) that also $K'(u_\epsilon)\phi < 0$. Hence for all small $\tau > 0$, $K(u_\epsilon + \tau\phi) < K(u_\epsilon) \leqslant \epsilon$. Thus, $u_\epsilon + \tau\phi \in \mathcal{C}_\epsilon$ for all small $\tau > 0$. Therefore, since $J'(u_\epsilon + u_0)\phi < 0$, we indeed get that $J(u_\epsilon + u_0 + \tau\phi) < J(u_\epsilon + u_0)$ for all small $\tau > 0$, a contradiction to (8). Therefore the claim $\mu_\epsilon \leqslant 0$ is true.

We now write (9) in its P.D.E. form as (with $g(s) = G'(s)$)

$$(P_\epsilon)\quad \begin{cases} -\Delta_N(u_\epsilon + u_0) + |u_\epsilon + u_0|^{N-2}(u_\epsilon + u_0) = f(x, u_\epsilon + u_0) + \mu_\epsilon g(u_\epsilon) & \text{in } \Omega, \\ |\nabla(u_\epsilon + u_0)|^{N-2}\dfrac{\partial(u_\epsilon + u_0)}{\partial\nu} = |u_\epsilon + u_0|^{q-1}(u_\epsilon + u_0) + \mu_\epsilon |u_\epsilon|^{\alpha-1}u_\epsilon & \text{on } \partial\Omega. \end{cases}$$

We now have two cases. **Case (i)**: $\inf_{\epsilon\in(0,1)} \mu_\epsilon > -\infty$, **Case (ii)**: $\inf_{\epsilon\in(0,1)} \mu_\epsilon = -\infty$.

In Case (i), we show that (up to a subsequence) $u_\epsilon \to 0$ in $W^{1,N}(\Omega)$. To see this, we define a new functional $I_\epsilon : W^{1,N}(\Omega) \to \mathbb{R}$ by $I_\epsilon(u) \stackrel{\text{def}}{=} J(u + u_0) - \mu_\epsilon K(u)$, $u \in W^{1,N}(\Omega)$, $\epsilon \in (0,1)$. Then, we see that using (9), $I'(u_\epsilon) = 0$, $\epsilon \in (0,1)$. Since $\{I_\epsilon(u_\epsilon)\}_{\epsilon\in(0,1)}$ is a bounded sequence (thanks to (7) and (8)) in $\mathbb{R}$, we may choose a subsequence (again denoted by $\{I_\epsilon(u_\epsilon)\}_{\epsilon\in(0,1)}$) such that $I_\epsilon(u_\epsilon) \to \rho$ as $\epsilon \to 0$. By the convexity of the dominating function in (f1), the constraint relation in (6) and Moser–Trudinger embedding, we get that $\{F(x, u_0 + u_\epsilon)\}_{\epsilon\in(0,1)}$ is a uniformly bounded sequence in $L^{3/2}(\Omega)$. Hence $\int_\Omega F(x, u_0 + u_\epsilon) \to \int_\Omega F(x, u_0)$ using Vitali's convergence theorem. Since $u_\epsilon \rightharpoonup 0$ in $W^{1,N}(\Omega)$, by Fatou's Lemma $J(u_0) \leqslant \rho$. Since $\rho = \lim_{\epsilon\to 0} J(u_\epsilon + u_0) \leqslant J(u_0)$ (from (8)), we obtain that $\rho = J(u_0)$. From the previous observation that $\int_\Omega F(x, u_0 + u_\epsilon) \to \int_\Omega F(x, u_0)$ and the obvious convergence $\int_{\partial\Omega} |u_\epsilon + u_0|^{q+1} \to \int_{\partial\Omega} |u_0|^{q+1}$, we obtain that $\|u_\epsilon\|_{W^{1,N}(\Omega)} \to 0$ as claimed before.

Hence, using the Trudinger–Moser type inequality in (4) we can apply Theorem 1.2 in Section 2 to conclude that $\sup_{\epsilon\in(0,1)} \|u_\epsilon\|_{L^\infty(\Omega)} \leqslant C$. Now, appealing to the regularity result of Lieberman [9], we get that $\sup_{\epsilon\in(0,1)} \|u_\epsilon\|_{C^{1,\mu}(\overline{\Omega})} < \infty$, for some $\mu \in (0,1)$.

We now consider the Case (ii) when $\mu_\epsilon \to -\infty$ as $\epsilon \to 0$. Now using (f2) and the fact that g is odd, we can find $M > 0$ independent of $\epsilon > 0$ and $x \in \overline{\Omega}$, such that $(f(x, u_0(x) + s) + \mu_\epsilon g(s))s$ and $(|u_0(x) + s|^{q-1}(u_0(x) + s) + \mu_\epsilon |s|^{\alpha-1}s)s$ are negative for all $s < -M$ and positive for all $s > M$. By the maximum principle (using $(u_\epsilon - M)^+$, $(u_\epsilon + M)^-$ as test functions), we get that $\sup_{\epsilon\in(0,1)} \|u_\epsilon\|_{L^\infty(\Omega)} \leqslant M$. We now let $\phi_\epsilon \stackrel{\text{def}}{=} |u_\epsilon|^{\beta-1}u_\epsilon$, $\beta > 1$, as a test function in (P_ϵ), integrate by parts and use the fact that $u \mapsto -\Delta_N u + |u|^{N-1}u$ is a monotone operator to get,

$$-\mu_\epsilon\Bigg[\int_\Omega g(u_\epsilon)|u_\epsilon|^{\beta-1}u_\epsilon + \int_{\partial\Omega} |u_\epsilon|^{\alpha+\beta}\Bigg] \leqslant \int_\Omega \big[f(x, u_0 + u_\epsilon) - f(x, u_0)\big]|u_\epsilon|^{\beta-1}u_\epsilon + \int_{\partial\Omega} \big[|u_0 + u_\epsilon|^{q-1}(u_0 + u_\epsilon) - |u_0|^{q-1}u_0\big]|u_\epsilon|^{\beta-1}u_\epsilon.$$

Hence, using the uniform $L^\infty(\Omega)$ estimate for $\{u_\epsilon\}_{\epsilon\in(0,1)}$ we get,

$$(-\mu_\epsilon)\Bigg[\int_\Omega g(u_\epsilon)|u_\epsilon|^{\beta-1}u_\epsilon + \int_{\partial\Omega} |u_\epsilon|^{\alpha+\beta}\Bigg] \leqslant C\Bigg(\int_\Omega |u_\epsilon|^\beta + \int_{\partial\Omega} |u_\epsilon|^\beta\Bigg).$$

Using the inequality $g(s)s \geqslant c|s|^{p+1}\ \forall s \in \mathbb{R}$, $\alpha \geqslant p$ and Hölder's we get,

$$(-\mu_\epsilon)\Bigg[\int_\Omega |u_\epsilon|^{p+\beta} + \int_{\partial\Omega} |u_\epsilon|^{p+\beta}\Bigg] \leqslant C\big(|\Omega|\big)\Bigg(\int_\Omega |u_\epsilon|^{p+\beta} + \int_{\partial\Omega} |u_\epsilon|^{p+\beta}\Bigg)^{\frac{\beta}{p+\beta}}.$$

Therefore, for any $\beta > 1$

$$(-\mu_\epsilon)\big[\|u_\epsilon\|^p_{L^{p+\beta}(\Omega)} + \|u_\epsilon\|^p_{L^{p+\beta}(\partial\Omega)}\big] \leqslant C\big(|\Omega|\big).$$

Letting $\beta \to \infty$ in the above equation we get,

$$\sup_{\epsilon\in(0,1)} (-\mu_\epsilon)\big(\|u_\epsilon\|^p_{L^\infty(\Omega)} + \|u_\epsilon\|^p_{L^\infty(\partial\Omega)}\big) \leqslant C\big(|\Omega|\big). \tag{10}$$

Using (10), the uniform L^∞ bounds for $\{u_\epsilon\}_{\epsilon\in(0,1)}$ in Ω as well as $\partial\Omega$ and the fact $g(s)|s|^{-p}$ is a function bounded below in $\mathbb{R}$, we get that the right-hand side terms in (P_ϵ) are uniformly bounded in Ω and $\partial\Omega$ for all $\epsilon \in (0,1)$. Then the standard regularity result of Lieberman [9] implies that $\sup_{\epsilon\in(0,1)} \|u_\epsilon\|_{C^{1,\mu}(\overline{\Omega})} < \infty$ for some $\mu \in (0,1)$. Thus, in either Case (i) or Case (ii), we obtain the uniform bound for $\{u_\epsilon\}_{\epsilon\in(0,1)}$ in $C^{1,\mu}(\overline{\Omega})$. This gives a contradiction to (2) since $u_\epsilon \to 0$ in $C^1(\overline{\Omega})$ and $J(u_0+u_\epsilon) < J(u_0)$, $\forall \epsilon > 0$ small. This contradiction proves the theorem. □

3. Uniform L^∞-bound for solutions of (P_ϵ)

Proof of Theorem 2.1. In what follows, C will denote a generic constant which may vary from equation to equation but is independent of $\epsilon \in (0,1)$. Consider the truncation functions $T_k(s) \overset{\text{def}}{=} (s+k)\chi_{(-\infty,-k]} + (s-k)\chi_{[k,+\infty)}$, for $k > 0$, which was introduced in Stampacchia [10]. Let $\Omega_k = \{x \in \Omega \mid |u_\epsilon| \geqslant k\}$, $\partial\Omega_k = \{x \in \partial\Omega \mid |u_\epsilon| \geqslant k\}$. We now take $T_k(u_\epsilon)$ as a test function in (P_ϵ) and (3) and use the fact $\mu_\epsilon \leqslant 0$ to get,

$$\int_\Omega \big(\big|\nabla(u_\epsilon+u_0)\big|^{N-2}\nabla(u_\epsilon+u_0) - |\nabla u_0|^{N-2}\nabla u_0\big)\cdot\nabla\big(T_k(u_\epsilon)\big) + \int_\Omega \big(|u_\epsilon+u_0|^{N-2}(u_\epsilon+u_0) - |u_0|^{N-2}u_0\big)T_k(u_\epsilon)$$
$$\leqslant \int_\Omega \big(f(x,u_\epsilon+u_0) - f(x,u_0)\big)T_k(u_\epsilon) + \int_{\partial\Omega} \big(|u_\epsilon+u_0|^{q-1}(u_\epsilon+u_0) - |u_0|^{q-1}u_0\big)T_k(u_\epsilon). \tag{11}$$

We now estimate from above the right-hand side of (11). Fix $\eta = \frac{N+1}{\theta-1}$, $r = \theta\eta$. Applying the generalised Hölder's inequality we get,

$$\text{R.H.S. of (11)} \leqslant \bigg(\int_\Omega \big(\big|f(x,u_\epsilon)\big| + \big|f(x,u_0)\big|\big)^\theta\bigg)^{\frac{1}{\theta}}\bigg(\int_\Omega \big|T_k(u_\epsilon)\big|^r\bigg)^{\frac{1}{r}}|\Omega_k|^{\frac{r-1-\eta}{r}} + \bigg(\int_{\partial\Omega} \big(|u_\epsilon|^q + |u_0|^q\big)^\theta\bigg)^{\frac{1}{\theta}}\bigg(\int_{\partial\Omega} \big|T_k(u_\epsilon)\big|^r\bigg)^{\frac{1}{r}}|\partial\Omega_k|^{\frac{r-1-\eta}{r}}$$
$$\leqslant C\bigg(\int_\Omega \big|T_k(u_\epsilon)\big|^r + \int_{\partial\Omega}\big|T_k(u_\epsilon)\big|^r\bigg)^{\frac{1}{r}}\big(|\partial\Omega_k| + |\partial\Omega_k|\big)^{\frac{r-1-\eta}{r}}. \tag{12}$$

In the last inequality, we made use of the trace embedding. We estimate from below the terms in the left-hand side of (11) using Sobolev and trace embeddings to get,

$$\text{L.H.S. of (11)} \geqslant C\bigg(\int_\Omega \big|\nabla\big(T_k(u_\epsilon)\big)\big|^N + \int_\Omega \big|T_k(u_\epsilon)\big|^N\bigg)$$
$$\geqslant C\bigg(\int_\Omega \big|T_k(u_\epsilon)\big|^r + \int_{\partial\Omega}\big|T_k(u_\epsilon)\big|^r\bigg)^{\frac{N}{r}}. \tag{13}$$

Now plugging the bounds in (12) and (13) into (11) we get,

$$\int_\Omega \big|T_k(u_\epsilon)\big|^r + \int_{\partial\Omega}\big|T_k(u_\epsilon)\big|^r \leqslant C\big(|\Omega_k| + |\partial\Omega_k|\big)^{\frac{N}{N-1}}. \tag{14}$$

We note that for $0 < k < h$, since $|T_k(s)| = (|s|-k)(1-\chi_{[-k,k]}(s))$, $\forall s \in \mathbb{R}$, and $\Omega_h \subset \Omega_k$,

$$\int_\Omega \big|T_k(u_\epsilon)\big|^r = \int_{\Omega_k} \big(|u_\epsilon| - k\big)^r \geqslant \int_{\Omega_h}\big(|u_\epsilon| - k\big)^r \geqslant (h-k)^r|\Omega_h|.$$

Similarly, $\int_{\partial\Omega} |T_k(u_\epsilon)|^r \geqslant (h-k)^r |\partial\Omega_h|$. Substituting the last two estimates in (14) and letting $\phi(k) \stackrel{\text{def}}{=} |\Omega_k| + |\partial\Omega_k|$, $k > 0$, we get,

$$\phi(h) \leqslant C(h-k)^{-r}\big(\phi(k)\big)^{\frac{N}{N-1}}, \quad 0 < k < h. \tag{15}$$

Let $d \stackrel{\text{def}}{=} 2^N C^{\frac{1}{r}}(|\Omega| + |\partial\Omega|)^{\frac{1}{(N-1)r}}$ and define a sequence $\{k_n\}$ by $k_0 = 0$ and

$$k_n = k_{n-1} + \frac{d}{2^n}, \quad n = 1, 2, \ldots. \tag{16}$$

Substituting (16) into (15) we get by induction

$$\phi(k_n) \leqslant \phi(0) 2^{nr(1-N)} \to 0 \quad \text{as } n \to \infty.$$

Since $\lim_{n\to\infty} k_n = d$ and ϕ is nonincreasing, we obtain $\phi(d) = |\Omega_d| + |\partial\Omega_d| = 0$. This implies,

$$\sup_{\epsilon\in(0,1)} \|u_\epsilon\|_{L^\infty(\Omega)} \leqslant d.$$

This proves Theorem 1.2. □

References

[1] A. Ambrosetti, H. Brezis, G. Cerami, Combined effects of concave and convex nonlinearities in some elliptic problems, J. Funct. Anal. 122 (1994) 519–543.

[2] H. Brezis, L. Nirenberg, Minima locaux relatifs à C^1 et H^1, C. R. Acad. Sci. Paris, Ser. I 317 (1993) 465–472.

[3] G. Azorero, J. Manfredi, I. Peral, Sobolev versus Hölder local minimizers and global multiplicity for some quasilinear elliptic equations, Commun. Contemp. Math. 2 (3) (2000) 385–404.

[4] F. Brock, L. Iturraga, P. Ubilla, A multiplicity result for the p-Laplacian involving a parameter, Ann. Inst. H. Poincaré Anal. Non Linéaire, in press.

[5] P. Cherrier, Meilleures constantes dans des inégalités relatives aux espaces de Sobolev, Bull. Sci. Math. 108 (1984) 225–262.

[6] P. Cherrier, Problèmes de Neumann nonlinéaires sur les variétés Riemanniennes, C. R. Acad. Sci. Paris, Sér. A 292 (1981) 637–640.

[7] D.G. De Figueiredo, J.P. Gossez, P. Ubilla, Local "superlinearity" and "sublinearity" for the p-Laplacian, in press.

[8] J. Giacomoni, S. Prashanth, K. Sreenadh, A global multiplicity result for N-Laplacian with critical nonlinearity of concave-convex type, J. Differential Equations 232 (2007) 544–572.

[9] G. Lieberman, Boundary regularity for solutions of degenerate elliptic equations, Nonlinear Anal. 12 (11) (1988) 1203–1219.

[10] G. Stampacchia, Equations Elliptiques du Second Ordre à Coefficients Discontinues, Les Presses de l'Université de Montréal, 1966.