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Bellman function and bilinear embedding theorem for Schrödinger-type operators

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Abstract

We discuss bilinear embedding theorems for a certain class of Schrödinger operators on L^p. The obtained estimates are dimension-free and linear in p. We outline a uniform proof of the theorem which relies on establishing three crucial properties of the concrete Bellman function we consider. ***To cite this article: O. Dragičević, A. Volberg, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***
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Résumé

Fonction de Bellman et le plongement bilinéaire pour des opérateurs de Schrödinger. On considère un théorème de plongement bilinéaire pour une classe des opérateurs de Schrödinger sur L^p. Le résultat ne depend pas de dimension et il est p-linéaire. On fait une esquisse de la démonstration basée sur trois observations concernant la fonction de Bellman spécifique. ***Pour citer cet article : O. Dragičević, A. Volberg, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***
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1. Introduction

For $n \in \mathbb{N}$ let $\Delta = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$ denote the usual Laplacian on $\mathbb{R}^n$. Let V be a non-negative function defined on finite real sequences and whose restriction to every $\mathbb{R}^n$ is measurable. By a slight abuse of notation we will use the same letter to denote *multiplication* with V. In this sense introduce formally the operator $L = -\Delta + V$, acting from $C_c^2(\mathbb{R}^n)$. We assume L admits a self-adjoint extension, also denoted by L; for a discussion on it see the monograph by Kato [5, V §5]. By means of the spectral theorem we can define operator semigroup generated by L. Let $\mathcal{K}_t$ be the heat kernel, i.e. the integral kernel associated to this semigroup. For precise definitions see [1, chapter 2].

We place the following conditions on V:

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(a) Kato's inequality:

$$\mathcal{K}_t(x,y) \leqslant C_1 t^{-\frac{n}{2}} e^{-\frac{a}{t}|x-y|^2}$$

and the kernel $\mathcal{K}_t$ is non-negative and uniformly integrable, i.e. for all $(x,t) \in \mathbb{R}^n \times (0,\infty)$,

$$\int_{\mathbb{R}^n} \mathcal{K}_t(x,y)\,\mathrm{d}y \leqslant 1.$$

(b) Gradient estimates for the heat kernel:

$$\left|\frac{\partial}{\partial x_j}\mathcal{K}_t(x,y)\right| \leqslant C_1 t^{-\frac{n+1}{2}} e^{-\frac{a}{t}|x-y|^2} \quad \text{and} \quad \left|\frac{\partial}{\partial t}\mathcal{K}_t(x,y)\right| \leqslant C_1 t^{-\frac{n}{2}-1} e^{-\frac{a}{t}|x-y|^2}.$$

(c) Let P_t denote the (Poisson) operator semigroup whose infinitesimal generator is $L^{1/2}$. If $g \in C_c^\infty$ then

$$\lim_{t\to 0} \int_{\mathbb{R}^n} \left| t\frac{\partial P_t g}{\partial t}(x)\right| \mathrm{d}x = 0 \quad \text{and} \quad \lim_{t\to\infty} \int_{\mathbb{R}^n} t\frac{\partial P_t g}{\partial t}(x)\,\mathrm{d}x = 0.$$

(d) For any bounded, non-negative, compactly supported function φ and some $C_2 > 0$ which does not depend on n,

$$\int_0^\infty \int_{\mathbb{R}^n} P_t\varphi(x) V(x)\,\mathrm{d}x\, t\,\mathrm{d}t \leqslant C_2 \|\varphi\|_1.$$

The conditions which imply (a) were studied in [7], for example. In general there exists a large literature on heat kernels and their estimates, e.g. [1]. The constants C_1, a from the conditions (a) and (b) are allowed to be dependent on n yet in Theorem 1.1 they still allow dimension-free estimates for general V as above.

For a given smooth $\mathbb{C}^N$-valued function $\phi = (\phi_1(x,t), \ldots, \phi_N(x,t))$ on $\mathbb{R}^n \times (0,\infty)$ denote

$$\|\phi\|_*^2 = \sum_{j=0}^n \left|\frac{\partial \phi}{\partial x_j}\right|^2 + V(x)\left|\phi(x,t)\right|^2,$$

where $x_0 = t$. This is the same as $\|\phi\|_*^2 = \|J\phi\|_{\mathrm{HS}}^2 + V(x)|\phi|^2$, where $J\phi$ is the Jacobi matrix of ϕ and $\|\cdot\|_{\mathrm{HS}}$ denotes the Hilbert–Schmidt norm. The L^p norm of a $\mathbb{C}^N$-valued test function ψ on $\mathbb{R}^n$ is of course $(\int_{\mathbb{R}^n} \|\psi(x)\|_{\mathbb{C}^N}^p\,\mathrm{d}x)^{1/p}$. Also, denote by q the conjugate exponent of p and $p^* = \max\{p,q\}$.

The next inequality is our main result; we call it the bilinear embedding theorem.

Theorem 1.1. *Let V satisfy properties* (a)–(d). *There is an absolute constant $C > 0$ such that for arbitrary natural numbers M, N, n, any pair $f : \mathbb{R}^n \to \mathbb{C}^M$ and $g : \mathbb{R}^n \to \mathbb{C}^N$ of C_c^∞ test functions and any $p > 1$ we have*

$$\int_0^\infty \int_{\mathbb{R}^n} \left\|P_t f(x)\right\|_* \left\|P_t g(x)\right\|_* \mathrm{d}x\, t\,\mathrm{d}t \leqslant C(p^*-1)\|f\|_p \|g\|_q.$$

The constant C only depends on the constant C_2 from the property (d).

Theorems of this type we already proved in [3] and [4], where they gave rise to dimension-free estimates of Riesz transforms associated to the Laplace, Ornstein–Uhlenbeck and Hermite operators. The constants were always of order $\mathrm{O}(p)$, which seems to be sharp in all of these cases. Thus we believe our method offers a unified way of proving sharp dimension-free L^p estimates for a vast array of Riesz transforms.

2. Bellman function

Here we introduce the function whose special properties are fundamental for our way of proving Theorem 1.1. These characteristics are stated in the theorem below; the proof was given in [4].

Throughout the section we assume that $p \geqslant 2$, $q = p/(p-1)$ and $\delta = q(q-1)/8$ are fixed. Observe that $\delta \sim (p-1)^{-1}$. Take $M, N \in \mathbb{N}$ and define

$$\Omega := \{(\zeta, \eta, Z, H) \in \mathbb{C}^M \times \mathbb{C}^N \times \mathbb{R} \times \mathbb{R};\ |\zeta|^p < Z, |\eta|^q < H\}.$$

Consider the function $B : \overline{\Omega} \to [0, \infty)$, defined as $B = \delta^{-1} Q$, where

$$Q(\zeta, \eta, Z, H) = 2(Z + H) - |\zeta|^p - |\eta|^q - \delta \widetilde{Q}(\zeta, \eta),$$

and

$$\widetilde{Q}(\zeta, \eta) = \begin{cases} |\zeta|^2 |\eta|^{2-q}, & |\zeta|^p \leqslant |\eta|^q, \\ \dfrac{2}{p}|\zeta|^p + \left(\dfrac{2}{q} - 1\right)|\eta|^q, & |\zeta|^p \geqslant |\eta|^q. \end{cases}$$

Function $\widetilde{Q}$ is in $C^1(\Omega)$ while its second derivatives only exist in the distributional sense. The Hessian matrix of Q, denoted by $d^2 Q$, is a matrix-valued function which maps vector $\omega \in \Omega$ into the matrix with entries $\frac{\partial^2 Q}{\partial\alpha\partial\beta}(\omega)$, where α and β range over $\zeta_j, \overline{\zeta_j}, \eta_k, \overline{\eta_k}, Z, H$ for $j = 1, \dots, M$, $k = 1, \dots, N$.

Theorem 2.1. *Let $\omega = (\zeta, \eta, Z, H) \in \Omega$. Then*

(i) $B(\omega) \leqslant 16(p-1)(Z + H)$.

There exists $\tau = \tau(|\zeta|, |\eta|) > 0$ such that

(ii) $-\mathrm{d}^2 B(\omega) \geqslant \tau |\mathrm{d}\zeta|^2 + \tau^{-1} |\mathrm{d}\eta|^2$;
(iii) $B(\omega) - \mathrm{d}B(\omega)\omega \geqslant \tau |\zeta|^2 + \tau^{-1} |\eta|^2$.

To clarify the notation let us say that by (ii) we mean that $\langle -\mathrm{d}^2 B(\omega) w, w\rangle \geqslant \tau |w_1|^2 + \tau^{-1} |w_2|^2$ for all $w = (w_1, w_2, w_3, w_4) \in \mathbb{C}^M \times \mathbb{C}^N \times \mathbb{R} \times \mathbb{R}$.

Function B stems from the work of F. Nazarov and S. Treil [6]. Their B was a function of four non-negative variables. For such B they proved another version of (ii), namely $-\mathrm{d}^2 B(\omega) \geqslant 2|\mathrm{d}\zeta||\mathrm{d}\eta|$. It turned out, as stated in Theorem 2.1, that this property can be strengthened and also carried without loss from the "scalar" to the "vector" case. But the most unexpected fact is that (iii) can be added to the list of properties of B. It appears Nazarov and Treil did not study anything like this, nor does (iii) seem to follow from (i) and (ii). It was thus a considerable surprise for us to see that (iii) is nevertheless also true, not the least since we can prove it with the same τ as in (ii), which is essential for applying B in the proof of Theorem 1.1.

The provenance of the need to study properties (ii) and especially (iii), crucial for successful uniform treatment of the operators with potential, is explained by the identity (1) below. The first such operator we studied was the Hermite operator [4], i.e. the case of $V(x) = |x|^2$. But it is clear from (1) that, owing to the properties of B, the lower estimates of Theorem 1.1 can be obtained regardless of the nature of V.

3. Sketch of the proof of Theorem 1.1

Let us briefly outline why we needed the assumptions (a)–(d) and how we merged them with Bellman function. Given test functions f, g on $\mathbb{R}^n$ we want to define $v(x, t) := (P_t f(x), P_t g(x), P_t|f|^p(x), P_t|g|^q(x))$ and furthermore $b := B \circ v$, that is,

$$b(x, t) := B(P_t f(x), P_t g(x), P_t|f|^p(x), P_t|g|^q(x)).$$

The existence of $\mathcal{K}_t(x, \cdot)$ as in (a) settles the definition of v. As for b, one has to ascertain that $v(x, t) \in \overline{\Omega}$. This again follows from (a).

Consider the operator L' defined for test functions on $\mathbb{R}^n \times (0,\infty)$ as $L' = L - \partial^2/\partial t^2$. Our aim is to estimate the integral

$$\int_0^\infty \int_{\mathbb{R}^n} L'b(x,t)\,\mathrm{d}x\,t\,\mathrm{d}t$$

from below and above. Note that $L'P_t\varphi = 0$. From here the chain rule immediately gives

$$L'b(x,t) = \sum_{j=0}^{n} \left\langle -\mathrm{d}^2 B(v_0)\frac{\partial v}{\partial x_j}(x,t), \frac{\partial v}{\partial x_j}(x,t)\right\rangle + V(x)\big[B(v_0) - \mathrm{d}B(v_0)v_0\big]. \tag{1}$$

Here we wrote $v_0 = v(x,t)$ and when $j = 0$ we meant the differentiation in t. Now the inequalities (ii) and (iii) quickly imply $L'b(x,t) \geqslant \|P_t f(x)\|_* \|P_t g(x)\|_*$. The formula (1) reveals, in retrospect, why those two properties of B were sought out in the first place.

As to the estimates from above, we first show that $\int \Delta b(x,t)\,\mathrm{d}x\,t\,\mathrm{d}t = 0$. This is done by means of the integration by parts, so that the integrals of $\partial^2 b/\partial x_j^2$ are reduced to those of $\partial b/\partial x_j$. To handle these terms we need to estimate x_j-derivatives of $\mathcal{K}_t(x,\cdot)$ and the ζ,η-derivatives of B. The former are supplied by (b), while the latter can be calculated explicitly:

$$\left|\frac{\partial B}{\partial \zeta}\right| \leqslant C(p)\max\big\{|\zeta|^{p-1}, |\eta|\big\} \quad \text{and} \quad \left|\frac{\partial B}{\partial \eta}\right| \leqslant C(p)|\eta|^{q-1}. \tag{2}$$

Thus we prove that $\int L'b = -\int \partial^2 b/\partial t^2 + \int V \cdot b$. The combination of (i) and (d) implies $\int V \cdot b$ is majorized by $C(p-1)(\|f\|_p^p + \|g\|_q^q)$. Finally, to estimate $\int \partial^2 b/\partial t^2$ we again integrate by parts. Four terms emerge, of which the non-trivial to estimate are $\liminf_{t\to 0} t \int_{\mathbb{R}^n} \partial b(x,t)/\partial t\,\mathrm{d}x$ and $-\limsup_{t\to\infty} t \int_{\mathbb{R}^n} \partial b(x,t)/\partial t\,\mathrm{d}x$. Recalling once again that $b = B \circ v$ and that v comprises of the extensions $P_t\varphi$, the estimates are reduced to the Poisson-kernel (and, by the subordination formula, the heat-kernel) estimates, i.e. precisely to (a)–(c).

To summarize, we get $\int \|P_t f(x)\|_* \|P_t g(x)\|_* \leqslant \int L'b \leqslant C(p-1)(\|f\|_p^p + \|g\|_q^q)$. Finally we replace f by λf, g by $\lambda^{-1} g$ and take minimum in $\lambda > 0$.

Remark 1. It seems feasible that the same Bellman function could be applied to obtain counterparts to the results in this paper but associated to the Laplacian on a Gaussian space. For a recent investigation of perturbations of the Gaussian Laplacian we refer the reader to [2].

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