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Universal Taylor series on arbitrary planar domains

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Abstract

Let $\Omega \subset \mathbb{C}$, $\Omega \neq \mathbb{C}$ be any domain and $\zeta \in \Omega$. Let $R = \mathrm{dist}(\zeta, \Omega^c) \in (0, +\infty)$ and $C(\zeta, R) = \{z \in \mathbb{C}\colon |\zeta - z| = R\}$. We set $J(\Omega, \zeta) = \Omega^c \cap C(\zeta, R)$. Then there exists $f \in H(\Omega)$, such that the sequence $S_N(f, \zeta)(z) = \sum_{n=0}^{N} \frac{f^{(n)}(\zeta)}{n!}(z-\zeta)^n$, $N = 0, 1, \ldots,$ approximates any polynomial uniformly on each compact set $K \subset J(\Omega, \zeta)$ with $\mathbb{C} \setminus K$ connected. This property of $f \in H(\Omega)$ is topologically and algebraically generic. ***To cite this article: V. Nestoridis, C. Papachristodoulos, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Séries universelles de Taylor sur des domaines planaires arbitraires. Soit $\Omega \subset \mathbb{C}$ un domaine, avec $\Omega \neq \mathbb{C}$. Soient aussi $\zeta \in \Omega$, $R = \mathrm{dist}(\zeta, \Omega^c) \in (0, +\infty)$ et $C(\zeta, R) = \{z \in \mathbb{C}\colon |\zeta - z| = R\}$. On pose $J(\Omega, \zeta) = \Omega^c \cap C(\zeta, R)$. Alors il existe $f \in H(\Omega)$ telle que la suite $S_N(f, \zeta)(z) = \sum_{n=0}^{N} \frac{f^{(n)}(\zeta)}{n!}(z - \zeta)^n$, $N = 0, 1, \ldots,$ approche tout polynôme uniformément sur tout compact $K \subset J(\Omega, \zeta)$ ne séparant pas le plan. Le phénomène est topologiquement et algébriquement générique. ***Pour citer cet article : V. Nestoridis, C. Papachristodoulos, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Version française abrégée

Les séries de Taylor universelles existent dans n'importe quel domaine simplement connexe, tandis qu'elles n'existent pas dans un anneau. La question qui se pose naturellement est la suivante : quelle sorte d'approximation est valable dans n'importe quel domaine $\Omega \subset \mathbb{C}$, avec $\Omega \neq \mathbb{C}$? On fixe $\zeta \in \Omega$ et on considère $R = \mathrm{dist}(\zeta, \Omega^c) \in (0, +\infty)$ et $C(\zeta, R) = \{z \in \mathbb{C}\colon |\zeta - z| = R\}$. On définit aussi $J(\Omega, \zeta) = \Omega^c \cap C(\zeta, R)$. On démontre alors qu'il existe $f \in H(\Omega)$ telle que la suite de ses sommes partielles $\sum_{k=0}^{n} \frac{f^{(k)}(\zeta)}{k!}(z - \zeta)^k$, $n = 0, 1, 2, \ldots,$ approche tout polynôme uniformément sur tout compact $K \subset J(\Omega, \zeta)$ qui ne sépare pas le plan. De plus, cette propriété est topologiquement et algébriquement générique. La preuve est essentiellement basée sur une combinaison du théorème de Runge et du théorème de Baire et étend des arguments de Costakis [6] et de Melas [23]. On établit aussi quelques propriétés de ces fonctions universelles et, comme application, on obtient une nouvelle preuve du fait que tout domaine de $\mathbb{C}$ est un do-

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maine d'holomorphie. En outre, on exhibe des exemples de séries trigonométriques universelles au sens de Menchoff et, sous quelques conditions supplémentaires, on obtient des informations sur la croissance des coefficients de Taylor et sur la sommabilité au sens de Césaro. Enfin, on remarque que ces résultats peuvent s'étendre aux développements de Faber.

1. Introduction

In the early 1970s W. Luh [19] and Chui and Parnes [10] showed the existence of universal Taylor series on any simply connected domain $\Omega \subset \mathbb{C}$ with respect to a center $\zeta \in \Omega$. More precisely, there is a holomorphic function $f \in H(\Omega)$, such that, the sequence of the partial sums of its Taylor development with center ζ, approximates any polynomial uniformly on any compact set $K \subset \mathbb{C}$ with connected complement which is disjoint from $\bar{\Omega}$. That is, we have approximation on subsets of $\bar{\Omega}^c$. In 1996 the first author [27,28] strengthened the previous result by allowing the compact set K to meet the boundary $\partial\Omega$; that is $K \cap \Omega = \emptyset$ and we have approximations on subsets of Ω^c. A consequence of the fact that the universal approximation is valid on the boundary also, is that the holomorphic function f has several wild properties ([1,3,4,6–9,11,15,16,18,22–25,27,28]; see also [12,13]). In [11] Gehlen, Luh and Müller showed that if the domain Ω is a bounded annulus, then there is no universal Taylor series and it is not possible to have such approximations on subsets of Ω^c nor of $\bar{\Omega}^c$. However, there are unbounded non-simply connected domains in $\mathbb{C}$, where universal Taylor series exist [6,32,23,2,31]. The following question arises naturally: Which approximation is possible for arbitrary planar domains Ω and is it topologically and algebraically a generic phenomenon? The purpose of the present Note is to give an answer to the previous question and establish some properties of the universal functions and some first consequences implied by such an approximation.

Let $\Omega \subset \mathbb{C}$, $\Omega \neq \mathbb{C}$ be any domain and fix a point $\zeta \in \Omega$. Then $R = \operatorname{dist}(\zeta, \Omega^c) = \operatorname{dist}(\zeta, \partial\Omega)$ is finite and strictly positive. We set $J(\Omega, \zeta)$ to be the intersection of Ω^c with the circle $\{z \in \mathbb{C}\colon |z - \zeta| = R\} = C(\zeta, R)$: $J(\Omega, \zeta) = \Omega^c \cap C(\zeta, R) = (\partial\Omega) \cap C(\zeta, R)$.

Obviously $J(\Omega, \zeta)$ is non void and compact. We prove that there exists a holomorphic function $f \in H(\Omega)$, such that the sequence of partial sums of its Taylor development with center ζ approximates every polynomial uniformly on each compact set $K \subset J(\Omega, \zeta)$ with $\mathbb{C} \setminus K$ connected. Further, the set of such f's is G_δ and dense in $H(\Omega)$ endowed with the topology of uniform convergence on compacta and contains a dense vector space except 0.

Combining with [18], the Taylor expansion of such f is a universal trigonometric series in the sense of Menchoff with respect to any σ-finite Borel measure on $J(\Omega, \zeta)$. In particular, if $J(\Omega, \zeta) = C(\zeta, R)$ (which is equivalent to $\Omega = \{z\colon |z - \zeta| < R\}$), then we obtain examples of the classical universal trigonometric series of Menchoff [26,27,14].

If $J(\Omega, \zeta)$ contains an arc of $C(\zeta, R)$ with strictly positive opening, then, as in [24], we obtain information on the growth of the Taylor coefficients. In particular, the sequence $R^n \frac{|f^{(n)}(\zeta)|}{n!}$, $n = 1, 2, \ldots$, cannot have polynomial growth. It follows that the Taylor expansion cannot be C_k summable for any $z \in J(\Omega, \zeta)$ and $k \geqslant 1$, provided that $J(\Omega, \zeta)$ contains an arc or even under weaker assumptions. Therefore, f cannot be continuously extendable on $\Omega \cup J(\Omega, \zeta)$.

Further, without any assumption on $J(\Omega, \zeta)$, it follows that the radius of convergence of the Taylor development of f with respect to ζ is exactly $R = \operatorname{dist}(\zeta, \partial\Omega)$. Varying ζ to a denumerable set E dense in Ω and applying Baire's Category theorem, we obtain that the generic function $f \in H(\Omega)$ satisfies that the radius of convergence of its Taylor expansion with center any $\zeta \in E$ is exactly $\operatorname{dist}(\zeta, \partial\Omega)$. Combining this with [29] it follows that f is non extendable and we have a new proof that every domain $\Omega \subset \mathbb{C}$ is a domain of holomorphy. The question that is naturally posed at this point is whether there exists $f \in H(\Omega)$, such that for every $\zeta \in \Omega$, the sequence of partial sums of its Taylor development with center ζ approximates every polynomial on every compact set $K \subset J(\Omega, \zeta)$ with connected complement. In the case of simply connected domains the answer is affirmative [20,28,25]; in the general case the question is open.

We also mention that similar generic results can further be obtained; in particular, for every polynomial p and every compact set $K \subset J(\Omega, \zeta)$ with connected complement, there is a sequence of partial sums $S_{\lambda_n}(f, \zeta)(z) = \sum_{k=0}^{\lambda_n} \frac{f^{(k)}(\zeta)}{k!} (z - \zeta)^k$ such that for every $\ell \in \{0, 1, 2, \ldots\}$ the sequence of the ℓ-derivatives $[S_{\lambda_n}(f, \zeta)]^{(\ell)}(z)$ converges to $p^{(\ell)}(z)$ uniformly on K. For this class we also have topological and algebraic genericity. The new class, where the approximation is required at the level of all derivatives, is a subclass of the initial one. We do not know if the inclusion is strict or not (see also [30]).

Finally we mention that similar results can be obtained, if we replace Taylor developments by Faber one's [17,21, 4,33]; then, the circle $C(\zeta, R)$ will be replaced by a level curve of a suitable conformal mapping.

2. Proof of the main result

Our main result is the following:

Theorem 2.1. *Let $\Omega \subset \mathbb{C}$, $\Omega \neq \mathbb{C}$ be any domain, let $\zeta \in \Omega$ be fixed and let μ be an infinite subset of $N = \{0, 1, 2, \ldots\}$. We set $J(\Omega, \zeta) = \Omega^c \cap C(\zeta, R)$, where $R = \operatorname{dist}(\zeta, \Omega^c) \in (0, +\infty)$ and $C(\zeta, R)$ denotes the circle with center ζ and radius R. Then there exists a holomorphic function $f \in H(\Omega)$ having the following property. For every compact set $K \subset J(\Omega, \zeta)$ with $\mathbb{C} \setminus K$ connected and every continuous function $h : K \to \mathbb{C}$, there exists a sequence $\lambda_n \in \mu$, $n = 1, 2, \ldots$, such that the sequence of partial sums $S_{\lambda_n}(f, \zeta)(z) = \sum_{k=0}^{\lambda_n} \frac{f^{(k)}(\zeta)}{k!}(z - \zeta)^k$ converges to h uniformly on K, as $n \to +\infty$. We denote by $B^\mu(\Omega, \zeta)$ the set of such f's. Then $B^\mu(\Omega, \zeta)$ is dense and G_δ in $H(\Omega)$ endowed with the topology of uniform convergence on compacta and contains a dense vector space except* 0.

For the proof we distinguish two cases: (i) $J(\Omega, \zeta) = C(\zeta, R)$ and (ii) $J(\Omega, \zeta) \neq C(\zeta, R)$. If $J(\Omega, \zeta) = C(\zeta, R)$, then one can easily see that $\Omega = \{z \in \mathbb{C}: |z - \zeta| < R\}$ and $B^\mu(\Omega, \zeta) \supset U^\mu(\Omega, \zeta)$, where $U^\mu(\Omega, \zeta)$ is the usual set of universal Taylor series in the disc Ω, according to [5]. Since $U^\mu(\Omega, \zeta) \neq \emptyset$ it follows $B^\mu(\Omega, \zeta) \neq \emptyset$. Because there exists a denumerable family of compact sets $K \subset C(\zeta, R)$ with $\mathbb{C} \setminus K$ connected, absorbing all other such sets, the Abstract Theory of universal series [5] can be applied in this case and the result follows.

From now on we consider that $J(\Omega, \zeta)$ is a proper compact subset of $C(\Omega, \zeta)$. Obviously $\mathbb{C} \setminus J(\Omega, \zeta)$ is connected and to prove the theorem, it suffices to consider only $K = J(\Omega, \zeta)$.

Let f_j, $j = 1, 2, \ldots$, be an enumeration of all the polynomials with coefficients in $Q + iQ$. Then the sequence f_j, $j = 1, 2, \ldots$, is dense in $C(J(\Omega, \zeta))$ and it suffices to consider only function $h = f_j$ for some $j = 1, 2, \ldots$.

For $j, s = 1, 2, \ldots$ and $n \in \{0, 1, 2, \ldots\}$ we consider the set $E(n, j, s) = \{g \in H(\Omega)\colon \sup_{z \in J(\Omega,\zeta)} |S_n(g, \zeta)(z) - f_j(z)| < \frac{1}{s}\}$ where $S_n(g, \zeta)(z) = \sum_{k=0}^{n} \frac{g^{(k)}(\zeta)}{k!}(z - \zeta)^k$. Then, as in [27], we have $B^\mu(\Omega, \zeta) = \bigcap_{j,s=1}^{\infty} \bigcup_{n \in \mu} E(n, j, s)$. Further, by Cauchy estimates, as in [27,28] the sets $E(n, j, s)$ are open in $H(\Omega)$. It follows that $B^\mu(\Omega, \zeta)$ is G_δ in $H(\Omega)$. Since $H(\Omega)$ is a complete metric space we can apply Baire's Category Theorem. Thus, if we prove that $\bigcup_{n \in \mu} E(n, j, s)$ is dense in $H(\Omega)$, then $B^\mu(\Omega, \zeta)$ is dense and G_δ in $H(\Omega)$.

Lemma 2.2. *For every $j, s = 1, 2, \ldots$ the set $\bigcup_{n \in \mu} E(n, j, s)$ is dense in $H(\Omega)$.*

Proof of the lemma. Let $f \in H(\Omega)$, $L \subset \Omega$ be compact and $\varepsilon > 0$. We are looking for $g \in H(\Omega)$ and $n \in \mu$, so that $\sup_{z \in L} |g(z) - f(z)| < \varepsilon$ and $g \in E(n, j, s)$.

Without loss of generality we may assume that every component of $(\mathbb{C} \cup \{\infty\}) \setminus L$ contains at least one component of $(\mathbb{C} \cup \{\infty\}) \setminus \Omega$ and that $\zeta \in L^\circ$. We fix a set A containing exactly one point from each component of $(\mathbb{C} \cup \{\infty\}) \setminus \Omega$.

If $a \in A$ satisfies $a \notin J(\Omega, \zeta)$ we set $\gamma(a) = a$.

If $a \in A$ satisfies $a \in J(\Omega, \zeta)$, then we consider W the component of $(\mathbb{C} \cup \{\infty\}) \setminus L$ containing a. Then $W \setminus J(\Omega, \zeta)$ is connected and open. Indeed, if $C(\zeta, R) \cap W \subset J(\Omega, \zeta)$, as $\partial W \subset L$, the distance of $J(\Omega, \zeta)$ from L would be zero, which is impossible as the compact sets L and $J(\Omega, \zeta)$ are disjoint. We choose $\gamma(a) \in W \setminus J(\Omega, \zeta)$ satisfying $|\gamma(a) - \zeta| > R$. We set $\Gamma = \{\gamma(a)\colon a \in A\}$. This set meets every component of $(\mathbb{C} \cup \{\infty\}) \setminus (L \cup J(\Omega, \zeta))$. We can find disjoint open sets V_1, V_2, $V_1 \cap V_2 = \emptyset$, $L \subset V_1 \subset \Omega$, $J(\Omega, \zeta) \subset V_2$. We consider the holomorphic function $F : V_1 \cup V_2 \to \mathbb{C}$ defined by $F_{|V_1} = f_{|V_1}$ and $F_{|V_2} = f_{j|V_2}$. By Runge's Approximation Theorem, there exists a rational function q with poles only in Γ such that

$$\sup_{z \in L \cup J(\Omega,\zeta)} \left|q(z) - F(z)\right| < \frac{\varepsilon'}{3}, \quad \text{where } \varepsilon' = \min\left(\varepsilon, \frac{1}{s}\right).$$

Then all poles of q have distance strictly greater than R from ζ; thus, as in [6], the Taylor development of q with center ζ converges uniformly on $J(\Omega, \zeta)$ and we find $n \in \mu$ so that $\sup_{z \in J(\Omega,\zeta)} |S_n(q, \zeta)(z) - q(z)| < \frac{\varepsilon'}{3}$. The integer $n \in \mu$ has now been fixed.

If all the poles $\gamma(a)$ of q belong to Ω^c, then we are done with $g = q$. If not, we observe that $\gamma(a)$ and a belong to the same component of $(\mathbb{C} \cup \{\infty\}) \setminus L$. If $\gamma(a) \in \Omega$, then $a \in J(\Omega, \zeta)$ and $a \in \Omega^c$. Thus, as in [23], we apply once more Runge's Theorem and find a rational function $g \in H(\Omega)$ such that $\sup_{z\in L}|g(z) - q(z)| < \eta$, where η, $0 < \eta < \frac{\varepsilon'}{3}$ will be chosen later. Since n is already fixed, $J(\Omega, \zeta)$ is bounded and $\zeta \in L^0$, by the Cauchy estimates we can choose η small enough so that $\sup_{z\in J(\Omega,\zeta)}|S_n(g, \zeta)(z) - S_n(q, \zeta)(z)| < \frac{\varepsilon'}{3}$. Thus we have found $n \in \mu$ and $g \in H(\Omega)$, so that $\sup_{z\in L}|f(z) - g(z)| < \varepsilon$ and $g \in E(n, j, s)$ and the proof of the lemma is completed. □

Since $B^\mu(\Omega, \zeta)$ is dense and G_δ in $H(\Omega)$ for every infinite subset $\mu \subset N$, the proof of 4) ⇒ 5) of Theorem 1 in [5] can be repeated and yields that $B^\mu(\Omega, \zeta)$ contains a dense vector space except 0. This completes the proof of our main result.

We note that the previous proof is not covered by the Abstract Theory [5] because the set of polynomials is not dense in $H(\Omega)$, if Ω is not simply connected.

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