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Asymptotic inversion of Toeplitz matrices with one singularity in the symbol

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Abstract

We consider the function f_α defined by $f_\alpha(\theta) = |1 - e^{i\theta}|^{2\alpha} f_1(e^{i\theta})$ with f_1 a regular strictly positive function and α a real number with $-\frac{1}{2} < \alpha$. For such a number α we compute the inverse $T_N(f_\alpha)^{-1}$ of the Toeplitz matrix $T_N(f_\alpha)$ and we obtain the asymptotic behaviour of the entries of this matrix when N goes to infinity. This inversion allows us to obtain two new families of kernels, H_α for $-\frac{1}{2} < \alpha < 0$, and G_α for $\alpha > 0$. We obtain also an asymptotic expansion of the coefficients of the orthogonal polynomials associated to the function f_α and we give an answer to a question of H. Kesten (1961). ***To cite this article: P. Rambour, A. Seghier, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Inversion asymptotique des matrices de Toeplitz dont le symbole présente une singularité. Considérons la fonction $f_\alpha(\theta) = |1 - e^{i\theta}|^{2\alpha} f_1(e^{i\theta})$ où f_1 est une fonction régulière strictement positive et α un nombre réel tel que $-\frac{1}{2} < \alpha$. Pour un tel α nous calculons l'inverse $T_N(f_\alpha)^{-1}$ de la matrice de Toeplitz $T_N(f_\alpha)$ et nous obtenons le comportement asymptotique des coefficients de cet inverse quand N tend vers l'infini. Ceci nous permet de mettre en évidence deux nouvelles familles de noyaux H_α pour $-\frac{1}{2} < \alpha < 0$, et G_α pour $\alpha > 0$. Nous obtenons également un développement asymptotique des coefficients des polynômes orthogonaux associés au poids f_α. Pour $\alpha = \frac{1}{2}$ nous répondons à une question énoncée par H. Kesten (1961). ***Pour citer cet article : P. Rambour, A. Seghier, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Version française abrégée

Si h est une fonction de $L^1(\mathbb{T})$, on note $T_N(h)$ la matrice $(N+1) \times (N+1)$ définie par $T_N(h)_{k+1,l+1} = (\hat{h}(k-l))$ où $\hat{h}(s)$ est le coefficient de Fourier d'ordre s de h. Dans la suite, si α est un réel strictement supérieur à $-\frac{1}{2}$, on pose $f_\alpha = |1-\chi|^{2\alpha} f_1$ en notant $\chi(e^{i\theta}) = e^{i\theta}$, et où f_1 est une fonction strictement positive sur le cercle unité. Nous supposons de plus que f_1 appartient à l'ensemble $\mathcal{C} = \{h\colon\ h > 0 \text{ et } h \in A(\mathbb{T}, \frac{3}{2})\}$ où $A(\mathbb{T}, \mu) = \{h \in C(\mathbb{T})\colon\ \|h\|_{A(\mathbb{T},\mu)} = \sum_{n\in\mathbb{Z}} |n+1|^{\mu} |\hat{h}(n)| < \infty\}$ [8]. Il est connu qu'on a alors les décompositions $f_1 = g_1 \bar{g}_1$, $f_\alpha = g_\alpha \bar{g}_\alpha$, avec g_1, $g_1^{-1} g_\alpha$,

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$g_\alpha^{-1} \in H^{2+}$. Nous supposons également $\frac{1}{2\pi}\int_0^{2\pi} \frac{1}{g_\alpha(t)}\,dt = 1$ ce qui ne réduit pas la généralité de nos énoncés. On obtient alors les Théorèmes 2.1 et 2.2. Il faut noter que le Théorème 2.2 donne l'asymptotique lorsque N tend vers l'infini des éléments $(T_N(|1-\chi|^{2\alpha} f_1))^{-1}_{[Nx]+1,[Ny]+1}$ pour $0 < x, y < 1$ et $-\frac{1}{2} < \alpha < 0$; par contre le Théorème 2.1 nous donne une expression asymptotique valable pour tout exposant α positif, ce qui nous permet de traiter un champs beaucoup plus large que H. Kesten dans [9] qui n'obtient ce résultat que pour $\alpha \in]\frac{1}{2}, 1[$. Dans ce même article H. Kesten propose une expression analytique pour $(T_N(|1-\chi|f_1))^{-1}_{[Nx]+1,[Ny]+1}$ mais ne peut rien dire sur le cas $x = y$. Les réponses à ces deux questions sont dans les Théorèmes 2.1 et 2.6. Dans le cas où α est un entier positif p le noyau G_p correspond au noyau de Green d'un opérateur différentiel d'ordre $2p$ [11,13].

Notre méthode est basée sur l'utilisation des polynômes prédicteurs de $|1-\chi|^{2\alpha} f_1$ dont les coefficients sont, à normalisation près, les éléments de la première colonne de $(T_N(|1-\chi|^{2\alpha} f_1))^{-1}$. Rappelons [10] que si P_M est le polynôme prédicteur de degré M d'une fonction h nous avons $\widehat{(\frac{1}{|P_M|^2})}(s) = \hat{h}(s)$, pour $-M \leqslant s \leqslant M$. On a alors $T_M(h) = T_M(\frac{1}{|P_M|^2})$ si P_M est le polynôme prédicteur de degré M de h, et également $Q_M(\chi) = \chi^N \overline{P_M(\chi)}$ si Q_M est le polynôme orthogonal de degré M associé au poids h. Notons $\sum_{u=0}^N \beta_{u,\alpha,N}\chi^u$ le polynôme prédicteur de degré N de $|1-\chi|^{2\alpha} f_1$. Dans une première étape nous calculons une asymptotique des $\beta_{u,\alpha,N}$ pour $\alpha \in]-\frac{1}{2}, \frac{1}{2}[$ avec une formule d'inversion adéquate (voir [12] et [15]), et nous les relions ensuite avec les coefficients du polynôme prédicteur de $|1-\chi|^{2\alpha}$ dont l'asymptotique est facile à déterminer (voir [5]). Puis par un passage à la limite nous obtenons les coefficients $\beta_{u,1/2,N}$. Une seconde étape consiste à obtenir les coefficients $\beta_{u,\alpha+1,N}$ à partir des $\beta_{u,\alpha,N}$ grâce à une formule de récursion établie dans [11,13]. Nous obtenons ainsi le Théorème 2.5. Enfin une formule à la Gohberg–Semencul [11,13] nous fournit l'asymptotique des coefficients de $(T_N(|1-\chi|^{2\alpha} f_1))^{-1}$ à partir des $\beta_{u,\alpha,N}$ pour tout réel $\alpha > -\frac{1}{2}$.

Application. En procédant de la même manière que H. Widom et H. Böttcher pour les exposants entiers [4] le noyau G_α permet d'obtenir une expression asymptotique quand N tend vers l'infini de la plus petite valeur propre de $(T_N(|1-\chi|^{2\alpha} f_1))^{-1}$ pour tout $\alpha > 0$.

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1. Introduction

Let h be a function belonging to $L^1(\mathbb{T})$. We denote by $T_N(h)$ the $(N+1)\times(N+1)$ Toeplitz matrix with $T_N(h)_{k+1,l+1} = (\hat{h}(k-l))$ constituted by the Fourier coefficients $\hat{h}(s)$ of h. It is well known that $T_N(h)$ is invertible for all $N \geqslant 0$ if h is positive. Let α be a real number in $\mathbb{R}\setminus\mathbb{Z}$ and $-\frac{1}{2} < \alpha$. In this paper we study the asymptotic behaviour of $(T_N(|1-\chi|^{2\alpha} f_1))^{-1}_{k+1,l+1}$ when N goes to infinity and where f_1 is a regular function that satisfies some natural hypothesis, and χ stands for $e^{i\theta}$. We say that f has a zero of fractionary order 2α if $\alpha > 0$ and a pole of fractionary order 2α if $\alpha < 0$.

The previous results about this asymptotic behaviour are those of H. Kesten [9] and P.M. Bleher [2]. For example H. Kesten [9] has obtained the first result, dealing with $\alpha \in]1/2, 1[$. The purpose of this work was to study random walks with characteristic function Φ such that $1 - \Phi = |1-\chi|^{2\alpha} f_1$. The second part of his paper was devoted to the applications to Toeplitz matrices, and he obtained the following asymptotic relation, for $0 \leqslant k \leqslant l \leqslant N$

$$f_1(1)\big(T_N\big(|1-\chi|^{2\alpha} f_1\big)\big)^{-1}_{k+1,l+1} = \frac{N^{2\alpha-1}}{\Gamma(\alpha)^2}|x-y|^{2\alpha-1}\int_0^{\kappa(x,y)} w^{\alpha-1}(1-w)^{-2\alpha}\,dw + o(1)$$

for

$$\kappa(x,y) = \frac{\min(x(1-x)^{-1}, y(1-y)^{-1})}{\max(x(1-x)^{-1}, y(1-y)^{-1})}$$

and $\lim_{N\to\infty} k/N = x$ and $\lim_{N\to\infty} l/N = y$ and $x \neq y$, $0 < x < 1$, $0 < y < 1$. For $x = y$ the expression in the right-hand side of the previous formula has to be read as

$$\frac{1}{f_1(1)\Gamma(\alpha)^2(2\alpha-1)} x^{2\alpha-1}(1-x)^{2\alpha-1}.$$

Moreover for $\alpha = \frac{1}{2}$ and $x \neq y$ Kesten said that the asymptotic expression of the entries of the matrix $(T_N(|1 - \chi|f_1))^{-1}$ is given by the right-hand side of the previous formula with $\alpha = \frac{1}{2}$, but he cannot conclude for $x = y$. The answers to these questions are in Theorems 2.1 and 2.6.

However in [2] P.M. Bleher obtained by analytical methods the coefficients $(T_N(|1 - \chi|^{2\alpha} f_1))^{-1}_{[Nx]+1,[Ny]+1}$ for $\frac{3}{4} < \frac{k}{N} < 1$ and $0 < \frac{l}{N} < \frac{1}{4}$ and $\alpha > 0$.

In previous works [11,13] we treated the case where α is a positive integer p, and we obtained

$$\big(T_N\big(|1-\chi|^{2p} f_1\big)\big)^{-1}_{[Nx]+1,[Ny]+1} = \frac{1}{((p-1)!)^2 f_1(1)} N^{2p-1} G_p(x,y) + o\big(N^{2p-1}\big) \tag{1}$$

where G_p is the Green kernel of a differential equation of order $2p$ with specific boundary conditions. It is a remarkable fact that for $\alpha > 0$ the kernel G_α has the same expression as the kernel G_p. The statement of the kernel G_p for an integer p required several steps (see [11] for the first version) and partly due H. Böttcher ([3], page 49 for the case where $f_1 = 1$). We obtain also an asymptotic expansion of the entries of the inverse for $\alpha = \frac{1}{2}$. This allows us to give a positive answer to a question by H. Kesten [9].

An application of the kernel G_α from Theorem 2.1 is to compute the extremal eigenvalues of the matrix $T_N(|1 - \chi|^{2\alpha} f_1)$, for all $\alpha > 0$. Indeed H. Widom and A. Böttcher [4] write an asymptotic relation when N goes to the infinity of the minimal eigenvalue of the Toeplitz matrix of symbol $h_p = |1 - \chi|^{2p} f_1$, for p a natural integer. Their result is that $\lambda_{\min}(T_N(h_p)) \sim \frac{c_p}{N^{2p}} f_1(1)$ where $c_p = \sqrt{8\pi p}(\frac{4p}{e})^{2p}(1 + O(\frac{1}{\sqrt{p}}))$. The work of these authors is closely related to the remark that $\frac{1}{\lambda_{\min}(T_N(h_p))} = \lambda_{\max}((T_N(h_p))^{-1})$ and that $\lambda_{\max}((T_N(h_p))^{-1}) = \frac{1}{f_1(1)} N^{2p} \lambda_{\max}(\mathcal{G}_p)$ where $\mathcal{G}_p$ is the integral operator of kernel G_p. Then they obtain the previous expression of $\lambda_{\min}(T_N(h_p))$. Using Theorem 2.1 we can extend this method to all matrices $T_N(|1 - \chi|^{2\alpha} f_1)$ with any positive real exponent α since the expression of G_α for any positive real α is the same as for the integers.

2. Main results

Let f_α be the function $f_\alpha(\theta) = |1 - \chi|^{2\alpha} f_1$. We assume that f_1 belongs to the set $\mathcal{C}$ defined by $\mathcal{C} = \{h\colon h > 0$ and $h \in A(\mathbb{T}, \frac{3}{2})\}$ were

$$A(\mathbb{T}, \mu) = \Big\{ h \in C(\mathbb{T})\colon \|h\|_{A(\mathbb{T},\mu)} = \sum_{n\in\mathbb{Z}} |n+1|^{\mu} |\hat{h}(n)| < \infty \Big\}$$

(see [8]). Then it is known that $f_1 = g_1 \bar{g}_1$, $g_1, g_1^{-1} \in H^{2+}$ and $f_\alpha = g_\alpha \bar{g}_\alpha$, $g_\alpha, g_\alpha^{-1} \in H^{2+}$. We put also $\frac{1}{g_\alpha} = \sum_{u\geqslant 0} \beta_u^\alpha \chi^u$. Without loss of generality we may assume that

$$\beta_0^\alpha = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{g_\alpha(t)} \, \mathrm{d}t = 1.$$

2.1. Asymptotic expansion of the entries of the inverse matrix

Theorem 2.1. *For $\alpha > 0$ and $0 < x, y < 1$, $x \neq y$, we obtain*

$$f_1(1)\big(T_N(f_\alpha)\big)^{-1}_{[Nx]+1,[Ny]+1} = N^{2\alpha-1} \frac{1}{\Gamma^2(\alpha)} \big(G_\alpha(x,y)\big) + o\big(N^{2\alpha-1}\big)$$

uniformly in (x,y) for $0 < \delta_1 \leqslant x, y \leqslant \delta_2 < 1$, $x \neq y$ where

$$G_\alpha(x,y) = x^\alpha y^\alpha \int_{\max(x,y)}^{1} \frac{(t-x)^{\alpha-1}(t-y)^{\alpha-1}}{t^{2\alpha}} \, \mathrm{d}t.$$

Theorem 2.2. *For $-\frac{1}{2} < \alpha < 0$, and $0 < x, y < 1$, $x \neq y$, we have*

$$f_1(1)\big(T_N(f_\alpha)\big)^{-1}_{[Nx]+1,[Ny]+1} = N^{2\alpha-1} \frac{1}{\Gamma^2(\alpha)} \big(H_\alpha(x,y)\big) + o\big(N^{2\alpha-1}\big)$$

uniformly in (x, y) *for* $0 < \delta_1 \leqslant x, y \leqslant \delta_2 < 1$, $x \neq y$ *where*

$$H_\alpha(x, y) = \lim_{\epsilon \to 0} x^\alpha y^\alpha \left(I_{\epsilon,\alpha}(x, y) - \frac{\epsilon^\alpha}{\alpha}(y - x)^{\alpha-1}(1 - y + x)^\alpha \right)$$

and

$$I_{\epsilon,\alpha}(x, y) = \int_{\max(x,y)(1+\epsilon\varphi(x,y))}^{1} \frac{(t - x)^{\alpha-1}(t - y)^{\alpha-1}}{t^{2\alpha}} \, \mathrm{d}t$$

and

$$\varphi(x, y) = \frac{1 - |x - y|}{\min(x, y)}.$$

A complete proof of Theorem 2.1 and Theorem 2.2 will be published in [15]. We can observe that $G_\alpha(x, x)$ and $H_\alpha(x, x)$ are not defined if $\alpha \in]-\frac{1}{2}, \frac{1}{2}[$. In [14] we give an asymptotic expansion of the entries $(T_N(f_\alpha))^{-1}_{k+1,k+1}$ for $0 \leqslant k \leqslant N$ and $-\frac{1}{2} < \alpha < \frac{1}{2}$, and this last result provides us an expression of $\mathrm{Tr}(T_N(f_\alpha))^{-1}$.

2.2. *Orthogonal polynomial, predictor polynomial*

Definition 2.3. Let $h \in L^1(\mathbb{T})$ be a non-negative function such $(T_N(h))^{-1}$ is defined for all integer N. Then the predictor polynomial of degree M of the function h is the polynomial $\sum_{u=0}^{M} \nu_u \chi^u$ with

$$\nu_u = \frac{1}{\mu_M}\big(T_M(h)\big)^{-1}_{u+1,1}, \quad \text{for } 0 \leqslant u \leqslant M, \quad \text{and} \quad \mu_M^2 = \big(T_M(h)\big)^{-1}_{1,1}.$$

It is well known [6] (see also [10] page 53), that the predictor polynomial verifies the following fundamental property:

Proposition 2.4. *If* P_M *is the predictor polynomial of degree* M *of* h *we have*

$$\widehat{\left(\frac{1}{|P_M|^2}\right)}(s) = \hat{h}(s), \quad -M \leqslant s \leqslant M,$$

where $\hat{t}(s)$ *denotes the* s*th Fourier coefficient of* t.

Remark 1. If P_N is the predictor polynomial of degree N of f, $T_N(f) = T_N(\frac{1}{|P_N|^2})$.

Remark 2. If Q_N is the is orthogonal polynomial of degree N associated with the weight h we have

$$Q_N(\chi) = \chi^N \overline{P_N(\chi)}.$$

We give now the asymptotic expansion of the coefficients of these polynomials.

Theorem 2.5. *If* $\alpha > -\frac{1}{2}$, $\alpha \neq 0$, *we obtain for* $0 < x < 1$,

$$g_1(1)\big(T_N\big(|1 - \chi|^{2\alpha} f_1\big)\big)^{-1}_{[Nx]+1,1} = N^{\alpha-1} \frac{1}{\Gamma(\alpha)} x^{\alpha-1}(1 - x)^\alpha + o\big(N^{\alpha-1}\big),$$

uniformly for x *in* $[\delta_1, \delta_2]$, $0 < \delta_1 < \delta_2 < 1$.

A particular but important case provides the asymptotic expression of $(T_N(f_\alpha))^{-1}_{k+1,1}$ for $-\frac{1}{2} < \alpha < 0$. This formula allows to compute the correlation coefficients for long memory processes (see [5] and [1]). These quantities also play an important role in prediction theory [7].

2.3. The case $\alpha = \frac{1}{2}$

Theorem 2.6. *For $\alpha = \frac{1}{2}$ we have*

$$f_1(1)\big(T_N\big(|1-\chi|f_1\big)\big)^{-1}_{k+1,k+1} = \frac{1}{\pi}\ln(N) + o(\ln N)$$

uniformly in k in $[\delta, 1-\delta]$, with $0 < \delta < \frac{1}{2}$.

In this case we have the following trace formula

$$f_1(1)\mathrm{Tr}\big(\big(T_N\big(|1-\chi|f_1\big)\big)^{-1}\big) = \frac{1}{\pi}N\ln N + o(N\ln N).$$

2.4. Trace Theorem

Theorem 2.7. *If $\alpha > \frac{1}{2}$, we have*

$$f_1(1)\mathrm{Tr}\big(\big(T_N\big(|1-\chi|^{2\alpha}f_1\big)\big)^{-1}\big) = N^{2\alpha}\frac{1}{\Gamma^2(\alpha)(2\alpha-1)}B(2\alpha,2\alpha) + o\big(N^{2\alpha}\big)$$

where $B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}t = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$.

For $-\frac{1}{2} < \alpha < \frac{1}{2}$ and $\alpha \neq 0$ this trace has also been obtained in the previous paper of the authors [14].

3. A sketch of the proof

The main idea of the proof is the observation that the necessary information on $(T_N(|1-\chi|^{2\alpha}f_1))^{-1}$ can be deduced from the formula for the predictor polynomial of degree N of $|1-\chi|^{2\alpha}f_1$. Denote it by

$$\sum_{u=0}^{N}\beta_{u,\alpha,N}\chi^u.$$

Using an adapted inversion formula, we compute an asymptotic expansion of the coefficients $\beta_{u,\alpha,N}$ (see [12] and [15]), which are closely related with the asymptotic expansion of the coefficients of the predictor polynomial of $|1-\chi|^{2\alpha}$. These last coefficients can be easily computed (see [5]). Then we prove that

$$\lim_{\alpha\to(1/2)^-}(\beta_{u,\alpha,N}) = \beta_{u,1/2,N}.$$

At the second step we use a recursive formula relating the coefficients $\beta_{u,\alpha,N}$ to the coefficients $\beta_{u,\alpha+1,N}$ (see [11,13]) to obtain Theorem 2.5.

At the last step, Gohberg–Semencul (see [11,13]) formula allows us to get the asymptotic expansion of the entries of $(T_N(f_\alpha))^{-1}$ from the coefficients $\beta_{u,\alpha,N}$.

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