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Mixed motives and the slice filtration

Pablo Pelaez

Département de mathématiques, Institut Galilée, Université Paris 13, 99, avenue Jean-Baptiste Clément, 93430 Villetaneuse, France

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Abstract

We construct several Quillen model structures in Jardine's category Spt of motivic symmetric T-spectra [J.F. Jardine, Motivic symmetric spectra, Doc. Math. 5 (2000) 445–553], such that their associated homotopy categories are naturally isomorphic to Voevodsky's slice filtration [V. Voevodsky, Open problems in the motivic stable homotopy theory. I, in: Motives, Polylogarithms and Hodge Theory, Part I, Int. Press Lect. Ser., Irvine, CA, 1998]. We prove a conjecture of Voevodsky [V. Voevodsky, Open problems in the motivic stable homotopy theory. I, in: Motives, Polylogarithms and Hodge Theory, Part I, Int. Press Lect. Ser., Irvine, CA, 1998], which says that over a perfect field all the slices s_q have a canonical structure of modules in Spt over the motivic Eilenberg–MacLane spectrum $H\mathbb{Z}$. Restricting the field even further to the case of characteristic zero, we get that the slices s_q may be interpreted as big motives in the sense of Voevodsky. We also show that the smash product in Spt induces pairings in the motivic Atiyah–Hirzebruch spectral sequence. ***To cite this article: P. Pelaez, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Motifs mixtes et la filtration par les tranches. Nous construisons plusieurs structures des modèles de Quillen dans la catégorie de Jardine Spt des T-spectres symétriques motiviques [J.F. Jardine, Motivic symmetric spectra, Doc. Math. 5 (2000) 445–553], tel que leur catégories d'homotopie associées sont naturellement isomorphiques à la filtration par les tranches de Voevodsky [V. Voevodsky, Open problems in the motivic stable homotopy theory. I, in: Motives, Polylogarithms and Hodge Theory, Part I, Int. Press Lect. Ser., Irvine, CA, 1998]. Nous prouvons une conjecture de Voevodsky [V. Voevodsky, Open problems in the motivic stable homotopy theory. I, in: Motives, Polylogarithms and Hodge Theory, Part I, Int. Press Lect. Ser., Irvine, CA, 1998], laquelle affirme que sur un corps parfait tous les tranches s_q sont canoniquement modules dans Spt sur le spectre motivique d'Eilenberg–MacLane $H\mathbb{Z}$. Si le corps est de charactéristique zéro, nous obtenons que les tranches s_q sont motifs grands au sens de Voevodsky. Nous montrons aussi que le produit « smash » dans Spt induit des structures multiplicatives sur la suite spectrale motivique de Atiyah–Hirzebruch. ***Pour citer cet article : P. Pelaez, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

1. Introduction

Let S be a Noetherian separated scheme of finite Krull dimension, $Sm|_S$ be the category of smooth schemes of finite type over S, Pre_S be the category of pointed simplicial presheaves on $Sm|_S$, and T in Pre_S be $S^1 \wedge \mathbb{G}_m$ where

E-mail address: pelaez@math.univ-paris13.fr.

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$\mathbb{G}_m$ is the multiplicative group $\mathbb{A}^1_S - \{0\}$ pointed by 1. Let Spt denote the category of symmetric T-spectra on Pre_S, $\mathrm{Spt}(\mathcal{M})$ denote Spt equipped with the motivic symmetric stable model structure constructed by Jardine [6], and $\mathcal{SH}$ its homotopy category, which is triangulated. We consider the following objects in $\mathrm{Spt}(\mathcal{M})$, $C^q_{\mathit{eff}} = \{F_n(S^r \wedge \mathbb{G}^s_m \wedge U_+) \mid n, r, s \geqslant 0;\ s - n \geqslant q;\ U \in Sm|_S\}$, where F_n is the left adjoint to the n-evaluation functor $ev_n : \mathrm{Spt} \to \mathrm{Pre}_S$. In order to get a motivic version of the Postnikov tower, Voevodsky [13] constructs a filtered family of triangulated subcategories of $\mathcal{SH}$, which we call the *slice filtration*:

$$\cdots \subseteq \Sigma^{q+1}_T \mathcal{SH}^{\mathit{eff}} \subseteq \Sigma^q_T \mathcal{SH}^{\mathit{eff}} \subseteq \Sigma^{q-1}_T \mathcal{SH}^{\mathit{eff}} \subseteq \cdots \tag{1}$$

where $\Sigma^q_T \mathcal{SH}^{\mathit{eff}}$ is the smallest full triangulated subcategory of $\mathcal{SH}$ which contains C^q_{eff} and is closed under arbitrary coproducts. The work of Neeman [9, Chapter 9], shows that the inclusion $i_q : \Sigma^q_T \mathcal{SH}^{\mathit{eff}} \hookrightarrow \mathcal{SH}$ has a right adjoint r_q, and that the functors $f_q, s_q : \mathcal{SH} \to \mathcal{SH}$ are exact, where $f_q = i_q r_q$, and for every X in $\mathrm{Spt}(\mathcal{M})$, $s_q(X)$ fits in the following distinguished triangle:

$$f_{q+1}X \xrightarrow{\rho^X_q} f_q X \xrightarrow{\pi^X_q} s_q X \longrightarrow \Sigma^{1,0}_T f_{q+1} X.$$

We say that $f_q(X)$ is the $(q-1)$-connective cover of X and $s_q(X)$ the q-slice of X.

2. Main results

Let A be a cofibrant ring spectrum with unit in $\mathrm{Spt}(\mathcal{M})$, and A-mod be the category of left A-modules in $\mathrm{Spt}(\mathcal{M})$. It follows directly from the work of Jardine [6, Proposition 4.19] and Hovey [5, Corollary 2.2] that the adjunction $(A \wedge -, U, \varphi) : \mathrm{Spt}(\mathcal{M}) \to A\text{-mod}$ induces a model structure $\mathrm{Spt}^A(\mathcal{M})$ in A-mod, i.e. a map f in $\mathrm{Spt}^A(\mathcal{M})$ is a weak equivalence or a fibration if and only if Uf is a weak equivalence or a fibration in $\mathrm{Spt}(\mathcal{M})$. In the rest of this Note $p, q \in \mathbb{Z}$ will denote arbitrary integers, and $u : \mathbf{1} \to A$ the unit map of A.

Theorem 2.1.

(i) *There exists a model structure* $R_{C^q_{\mathit{eff}}} \mathrm{Spt}(\mathcal{M})$ *in* Spt *such that its homotopy category* $R_{C^q_{\mathit{eff}}} \mathcal{SH}$ *is triangulated and naturally equivalent as a triangulated category to* $\Sigma^q_T \mathcal{SH}^{\mathit{eff}}$. *Furthermore, the identity* $id : \mathrm{Spt}(\mathcal{M}) \to R_{C^q_{\mathit{eff}}} \mathrm{Spt}(\mathcal{M})$ *is a right Quillen functor, and the functor* f_q *is canonically isomorphic to the following composition of exact functors*:

$$\mathcal{SH} \xrightarrow{R} R_{C^q_{\mathit{eff}}} \mathcal{SH} \xrightarrow{C_q} \mathcal{SH} \tag{2}$$

where R *denotes a fibrant replacement functor in* $\mathrm{Spt}(\mathcal{M})$*, and* C_q *denotes a cofibrant replacement functor in* $R_{C^q_{\mathit{eff}}} \mathrm{Spt}(\mathcal{M})$.

(ii) *There exists a model structure* $S^q \mathrm{Spt}(\mathcal{M})$ *in* Spt *such that its homotopy category* $S^q \mathcal{SH}$ *is triangulated and the identity* $id : R_{C^q_{\mathit{eff}}} \mathrm{Spt}(\mathcal{M}) \to S^q \mathrm{Spt}(\mathcal{M})$ *is a left Quillen functor. Furthermore, the functor* s_q *is canonically isomorphic to the following composition of exact functors*:

$$\mathcal{SH} \xrightarrow{R} R_{C^q_{\mathit{eff}}} \mathcal{SH} \xrightarrow{C_q} S^q \mathcal{SH} \xrightarrow{W_{q+1}} R_{C^q_{\mathit{eff}}} \mathcal{SH} \xrightarrow{C_q} \mathcal{SH} \tag{3}$$

where W_{q+1} *denotes a fibrant replacement functor in* $S^q \mathrm{Spt}(\mathcal{M})$. *In the rest of this note* f_q, s_q *will denote respectively* $C_q \circ R$, $C_q \circ W_{q+1} \circ C_q \circ R$.

(iii) *The smash product in* Spt *induces the following Quillen bifunctors in the sense of Hovey* [3]:

$$R_{C^p_{\mathit{eff}}} \mathrm{Spt}(\mathcal{M}) \times R_{C^q_{\mathit{eff}}} \mathrm{Spt}(\mathcal{M}) \xrightarrow{-\wedge-} R_{C^{p+q}_{\mathit{eff}}} \mathrm{Spt}(\mathcal{M}),$$

$$S^p \mathrm{Spt}(\mathcal{M}) \times S^q \mathrm{Spt}(\mathcal{M}) \xrightarrow{-\wedge-} S^{p+q} \mathrm{Spt}(\mathcal{M}). \tag{4}$$

(iv) *The homotopy category* $\mathcal{SH}^A$ *of* $\mathrm{Spt}^A(\mathcal{M})$ *is triangulated. Furthermore, there exist model structures* $R_{C^q_{\mathit{eff}}}\mathrm{Spt}^A(\mathcal{M})$, $S^q\,\mathrm{Spt}^A(\mathcal{M})$ *in* A-mod *such that their homotopy categories* $R_{C^q_{\mathit{eff}}}\mathcal{SH}^A$, $S^q\mathcal{SH}^A$ *are triangulated, and the identity functors* $\mathrm{Spt}^A(\mathcal{M}) \xleftarrow{id} R_{C^q_{\mathit{eff}}}\mathrm{Spt}^A(\mathcal{M}) \xrightarrow{id} S^q\,\mathrm{Spt}^A(\mathcal{M})$ *are left Quillen functors. We will denote by* f_q^A, s_q^A *the following compositions of exact functors*:

$$\begin{aligned}&\mathcal{SH}^A \xrightarrow{R^A} R_{C^q_{\mathit{eff}}}\mathcal{SH}^A \xrightarrow{C_q^A} \mathcal{SH}^A,\\ &\mathcal{SH}^A \xrightarrow{R^A} R_{C^q_{\mathit{eff}}}\mathcal{SH}^A \xrightarrow{C_q^A} S^q\mathcal{SH}^A \xrightarrow{W_{q+1}^A} R_{C^q_{\mathit{eff}}}\mathcal{SH}^A \xrightarrow{C_q^A} \mathcal{SH}^A\end{aligned} \tag{5}$$

where C_q^A *denotes a cofibrant replacement functor in* $R_{C^q_{\mathit{eff}}}\mathrm{Spt}^A(\mathcal{M})$; *and* R^A, W_{q+1}^A *denote fibrant replacement functors respectively in* $\mathrm{Spt}^A(\mathcal{M})$, $S^q\,\mathrm{Spt}^A(\mathcal{M})$.

(v) *If* A *is cofibrant in* $R_{C^0_{\mathit{eff}}}\mathrm{Spt}(\mathcal{M})$, *then the exact functors* $f_q \circ UR^A, s_q \circ UR^A : \mathcal{SH}^A \to \mathcal{SH}$ *factor* (*up to a canonical isomorphism*) *through* $\mathcal{SH}^A$ *as follows*:

$$\begin{array}{ccc} \mathcal{SH}^A & \xrightarrow{UR^A} & \mathcal{SH} \\ {\scriptstyle f_q^A}\downarrow & & \downarrow{\scriptstyle f_q} \\ \mathcal{SH}^A & \xrightarrow{UR^A} & \mathcal{SH} \end{array} \qquad \begin{array}{ccc} \mathcal{SH}^A & \xrightarrow{UR^A} & \mathcal{SH} \\ {\scriptstyle s_q^A}\downarrow & & \downarrow{\scriptstyle s_q} \\ \mathcal{SH}^A & \xrightarrow{UR^A} & \mathcal{SH} \end{array} \tag{6}$$

This means that for every A*-module* M*, its* $(q-1)$*-connective cover* $f_q(M)$ *and* q*-slice* $s_q(M)$ *inherit a natural structure of* A*-module in* $\mathrm{Spt}(\mathcal{M})$.

(vi) *If* A *is cofibrant in* $R_{C^0_{\mathit{eff}}}\mathrm{Spt}(\mathcal{M})$ *and its unit map* u *is a weak equivalence in* $S^0\,\mathrm{Spt}(\mathcal{M})$, *then the functor* $s_q : \mathcal{SH} \to \mathcal{SH}$ *is canonically isomorphic to the following composition of exact functors*:

$$\mathcal{SH} \xrightarrow{R} R_{C^q_{\mathit{eff}}}\mathcal{SH} \xrightarrow{C_q} S^q\mathcal{SH} \xrightarrow{A\wedge P_q -} S^q\mathcal{SH}^A \xrightarrow{W_{q+1}^A} R_{C^q_{\mathit{eff}}}\mathcal{SH}^A \xrightarrow{C_q^A} \mathcal{SH}^A \xrightarrow{UR^A} \mathcal{SH} \tag{7}$$

where P_q *denotes a cofibrant replacement functor in* $S^q\,\mathrm{Spt}(\mathcal{M})$. *This means that for every* X *in* $\mathrm{Spt}(\mathcal{M})$, *its* q*-slice* $s_q(X)$ *is naturally equipped with a structure of* A*-module in* $\mathrm{Spt}(\mathcal{M})$.

Sketch of the proof. The model category $R_{C^q_{\mathit{eff}}}\mathrm{Spt}(\mathcal{M})$ is defined as the right Bousfield localization of $\mathrm{Spt}(\mathcal{M})$ with respect to the C^q_{eff}-colocal equivalences. On the other hand, $S^q\,\mathrm{Spt}(\mathcal{M})$ is defined as a right Bousfield localization with respect to an auxiliary model structure $L_{<q+1}\mathrm{Spt}(\mathcal{M})$, which is a left Bousfield localization of $\mathrm{Spt}(\mathcal{M})$ and its fibrant replacement functor describes the cone of the map $\tau_q^X : f_qX \to X$. The model structures $R_{C^q_{\mathit{eff}}}\mathrm{Spt}^A(\mathcal{M})$, $S^q\,\mathrm{Spt}^A(\mathcal{M})$ are constructed similarly using the left adjoint $A \wedge -$ to define the maps that get inverted in the Bousfield localization. To show that the smash product in Spt induces Quillen bifunctors, we use Hovey's approach to symmetric monoidal model categories [3, Chapter 4], and the triangulated structure in the homotopy categories $R_{C^q_{\mathit{eff}}}\mathcal{SH}$, $S^q\mathcal{SH}$. The Bousfield localizations are constructed following Hirschhorn's approach [2]. In order to apply Hirschhorn's techniques, it is necessary to check that $\mathrm{Spt}(\mathcal{M})$ and $\mathrm{Spt}^A(\mathcal{M})$ are both *cellular*; for this we rely on Hovey's general approach to spectra [4] and on an unpublished result of Hirschhorn [10, Theorem 2.2.4].

3. Applications

Using the slice filtration (1), it is possible to construct a spectral sequence which is an analogue of the classical Atiyah–Hirzebruch spectral sequence in algebraic topology. Namely, let X, Y be in $\mathrm{Spt}(\mathcal{M})$, and $[-,-]$ be the set of maps between two objects in $\mathcal{SH}$. Then the collection of distinguished triangles $\{f_{q+1}X \to f_qX \to s_qX \to \Sigma_T^{1,0} f_{q+1}X\}$ generates an exact couple $(D_1^{p,q}(Y;X), E_1^{p,q}(Y;X))$, where $D_1^{p,q}(Y;X) = [Y, \Sigma_T^{p+q,0} f_pX]$ and $E_1^{p,q}(Y;X) = [Y, \Sigma_T^{p+q,0} s_pX]$.

Theorem 3.1. *Let X, X', Y, Y' be in* $\mathrm{Spt}(\mathcal{M})$. *Then the smash product in* $\mathrm{Spt}(\mathcal{M})$ *induces natural external pairings in the motivic Atiyah–Hirzebruch spectral sequence*:

$$E_r^{p,q}(Y;X)\otimes E_r^{p',q'}(Y';X')\longrightarrow E_r^{p+p',q+q'}(Y\wedge Y';X\wedge X').$$

Theorem 2.1(i)–(iii) and [3, Theorem 1.4.3] imply that the smash product in Spt induces natural pairings $\bigcup^c_{p,q}: f_p-\wedge f_q-\to f_{p+q}-$, $\bigcup^s_{p,q}: s_p-\wedge s_q-\to s_{p+q}-$ in $\mathcal{SH}$. Furthermore, the following diagrams are commutative in $\mathcal{SH}$ (cf. [10, Theorem 3.6.10])

$$\begin{array}{ccccc} f_{p+1}X\wedge f_qY & \xrightarrow{\rho_p^X\wedge id} & f_pX\wedge f_qY & \xrightarrow{\pi_p^X\wedge\pi_q^Y} & s_pX\wedge s_qY \\ \downarrow{\scriptstyle \bigcup^c_{p+1,q}} & & \downarrow{\scriptstyle \bigcup^c_{p,q}} & & \downarrow{\scriptstyle \bigcup^s_{p,q}} \\ f_{p+q+1}(X\wedge Y) & \xrightarrow{\rho_{p+q}^{X\wedge Y}} & f_{p+q}(X\wedge Y) & \xrightarrow{\pi_{p+q}^{X\wedge Y}} & s_{p+q}(X\wedge Y) \end{array}\qquad \begin{array}{ccc} f_pX\wedge f_{q+1}Y & \xrightarrow{id\wedge\rho_q^Y} & f_pX\wedge f_qY \\ \downarrow{\scriptstyle \bigcup^c_{p,q+1}} & & \downarrow{\scriptstyle \bigcup^c_{p,q}} \\ f_{p+q+1}(X\wedge Y) & \xrightarrow{\rho_{p+q}^{X\wedge Y}} & f_{p+q}(X\wedge Y) \end{array}$$

Hence, the result follows from the work of Massey [8], and Proposition 14.3 in [1].

The following theorem proves a conjecture of Voevodsky [13]:

Theorem 3.2. *Let $H\mathbb{Z}$ denote the motivic Eilenberg–MacLane spectrum in* $\mathrm{Spt}(\mathcal{M})$ (*cf.* [7, Example 8.2.2(2)]). *If the base scheme S is a perfect field, then for every X in* $\mathrm{Spt}(\mathcal{M})$, *its q-slice $s_q(X)$ has a natural structure of $H\mathbb{Z}$-module in* $\mathrm{Spt}(\mathcal{M})$.

The work of Voevodsky [12] (for characteristic zero) and Levine [7, Lemma 10.4.1 and Theorem 10.5.1] (in general) shows that $s_0(H\mathbb{Z})\cong H\mathbb{Z}$ in $\mathcal{SH}$ and $s_0(u)$ is a weak equivalence in $\mathrm{Spt}(\mathcal{M})$, where u denotes the unit map of $H\mathbb{Z}$. Thus, it follows from [10, Lemma 3.6.21] that $H\mathbb{Z}$ is cofibrant in $R_{C^0_{eff}}\,\mathrm{Spt}(\mathcal{M})$ and u is a weak equivalence in $S^0\,\mathrm{Spt}(\mathcal{M})$. Hence, the result is an immediate consequence of Theorem 2.1(vi).

If we restrict the field even further to the case of characteristic zero, then we get that the slices may be interpreted as motives in the sense of Voevodsky.

Theorem 3.3. *If k is a field of characteristic zero, then for every X in* $\mathrm{Spt}(\mathcal{M})$ *its q-slice $s_q(X)$ is a big motive in the sense of Voevodsky* (*cf.* [11, Section 2.3]).

This follows from [11, Theorem 1.1] and Theorem 3.2.

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