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Sobolev weak solutions for parabolic PDEs and FBSDEs ☆

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Abstract

This Note is devoted to the representation of Sobolev weak solutions to quasi-linear parabolic PDEs with monotone coefficients via FBSDEs. One distinctive character of this result is that the forward component of the FBSDE is coupled with the backward variable. ***To cite this article: F. Zhang, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Solutions faibles de Sobolev des EDP paraboliques et des EDSR. Cette Note est consacré à la représentation des solutions faibles au sens de Sobolev d'une EDP quasi-linéaire parabolique avec des coefficients monotones par un système d'EDS et d'EDSR. Un caractère distinctif de ce résultat est que la composante de l'EDS est couplée avec les solutions d'EDSR. ***Pour citer cet article : F. Zhang, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

1. Introduction

In this Note, we consider the following quasi-linear parabolic partial differential equation (PDE for short):

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) + \mathcal{L}u(t,x) + g(t,x,u(t,x),(\sigma^*\nabla u)(t,x)) = 0 \quad \forall (t,x) \in [0,T) \times \mathbb{R}^d, \\ u(T,x) = h(x) \quad \forall x \in \mathbb{R}^d, \end{cases} \tag{1}$$

where $\mathcal{L}u = \sum_{i=1}^d b_i(t,x,u)\frac{\partial u}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^d a_{i,j}\frac{\partial^2 u}{\partial x_i \partial x_j}$, $(b_1,\dots,b_d)^* = b$, $(a_{i,j}) = \sigma\sigma^*$. It is worth noting that the coefficient b depends on the solution variable u. We will conclude that PDE (1) admits a unique weak solution in the Sobolev spaces. The method consists in relating it to the following forward backward stochastic differential equation (FBSDE):

$$\begin{cases} X_s^{t,x} = x + \int_t^s b(r, X_r^{t,x}, Y_r^{t,x})\,\mathrm{d}r + \int_t^s \sigma(r, X_r^{t,x})\,\mathrm{d}W_r, \\ Y_s^{t,x} = h(X_T^{t,x}) + \int_s^T g(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})\,\mathrm{d}r - \int_s^T Z_r^{t,x}\cdot \mathrm{d}W_r. \end{cases} \tag{2}$$

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FBSDEs can be used to give the probabilistic interpretation of certain quasi-linear parabolic PDEs. Under smooth coefficients assumptions the PDE has a classical solution. However, if its coefficients are not smooth enough, the PDE has to be solved in a weak way. Two notions of weak solutions can be used for the PDE. One is the viscosity solution, which was first introduced by Crandall and Lions for first-order Hamilton–Jacobi equations. This link with FBSDEs has been done by Peng [7], Pardoux and Peng [5], Pardoux and Tang [6]. Another way of defining weak solutions of PDEs, which is more classical, is the notion of Sobolev weak solutions. Barles and Lesigne [3] were the first to use this approach to connect quasi-linear PDEs with FBSDEs. Bally and Matoussi [2] studied the solution in Sobolev spaces of the quasi-linear stochastic PDE in terms of the backward doubly stochastic differential equation (BDSDE) with Lipschitz coefficients. Ouknine and Turpin [4] studied the Sobolev weak solutions of parabolic PDEs under weaker assumptions.

It is worth noting that, unlike the papers in which the viscosity solutions of PDEs were studied, to the author's knowledge, in all literature related to the Sobolev weak solutions of parabolic PDEs via FBSDEs, the forward equation in the FBSDE is completely decoupled from the backward one. Our paper is the first attempt to consider the Sobolev solutions of parabolic PDEs via FBSDEs whose forward component is coupled with the backward one. To be precise, the coefficient b depends on the variable y.

In the second part of this Note, some notations and known results about Sobolev weak solutions are given. In the third part, we state our main result, which gives the unique Sobolev weak solution of PDE (1) through the solution of FBSDE (2).

2. Preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $T > 0$ and $\{W_t, 0 \leqslant t \leqslant T\}$ a d-dimensional standard Brownian motion. Let $\mathcal{F}_t^s$ be the filtration generated by $\{W_r - W_t, t \leqslant r \leqslant s\}$, augmented with the $\mathbb{P}$-null sets of $\mathcal{F}$. For $n \in \mathbb{N}$, let $M^2(t, T; \mathbb{R}^n)$ denote the set of n-dimensional $\mathcal{F}_t^s$-progressively measurable processes $\{\varphi_s, t \leqslant s \leqslant T\}$ such that $E\int_t^T |\varphi_s|^2\,\mathrm{d}t < \infty$. Similarly, let $S^2(t, T; \mathbb{R}^n)$ denote the set of n-dimensional $\mathcal{F}_t^s$-progressively measurable continuous processes $\{\varphi_s, t \leqslant s \leqslant T\}$ such that $E\sup_{t\leqslant s\leqslant T} |\varphi_s|^2 < \infty$.

Suppose $\rho : \mathbb{R}^d \to \mathbb{R}^+$ is a continuous function such that $\int_{\mathbb{R}^d}(1 + |x|^2)\rho(x)\,\mathrm{d}x < \infty$ and $L^2(\mathbb{R}^d, \rho(x)\,\mathrm{d}x)$ is the space of $\varphi : \mathbb{R}^d \to \mathbb{R}$ such that $\int_{\mathbb{R}^d} |\varphi(x)|^2\rho(x)\,\mathrm{d}x < \infty$. We denote by $\mathcal{H}$ the set of $\{u(t, x), 0 \leqslant t \leqslant T, x \in \mathbb{R}^d\}$ such that u, $\sigma^*\nabla u$ belong to $L^2((0, T) \times \mathbb{R}^d; \mathrm{d}s \otimes \rho(x)\,\mathrm{d}x)$. $\mathcal{H}$ is a Banach space endowed with the norm $\|u\|_{\mathcal{H}}^2 := \int_{\mathbb{R}^d}\int_t^T (|u(s, x)|^2 + |(\sigma^*\nabla u)(s, x)|^2)\,\mathrm{d}s\,\rho(x)\,\mathrm{d}x$. Here and in the sequel, the derivatives are understood in the distribution sense.

Definition 1. We say that u is a Sobolev weak solution of PDE (1) if $u \in \mathcal{H}$ and for any $\varphi \in \mathcal{C}_c^{1,\infty}([0, T] \times \mathbb{R}^d)$,

$$
\begin{aligned}
&\int_{\mathbb{R}^d}\int_t^T u(s, x)\partial_s\varphi(s, x)\,\mathrm{d}s\,\mathrm{d}x + \int_{\mathbb{R}^d} u(t, x)\varphi(t, x)\,\mathrm{d}x - \int_{\mathbb{R}^d} h(x)\varphi(T, x)\,\mathrm{d}x \\
&\quad - \frac{1}{2}\int_{\mathbb{R}^d}\int_t^T (\sigma^*\nabla u)(s, x)\cdot(\sigma^*\nabla\varphi)(s, x)\,\mathrm{d}s\,\mathrm{d}x - \int_{\mathbb{R}^d}\int_t^T u\,\mathrm{div}\big((b - A)\varphi\big)(s, x)\,\mathrm{d}s\,\mathrm{d}x \\
&= \int_{\mathbb{R}^d}\int_t^T g\big(s, x, u(s, x), (\sigma^*\nabla u)(s, x)\big)\varphi(s, x)\,\mathrm{d}s\,\mathrm{d}x,
\end{aligned}
$$

where $A = (A_1, \ldots, A_d)^*$, $A_j = \frac{1}{2}\sum_{i=1}^d \frac{\partial a_{ij}}{\partial x_i}$, $1 \leqslant j \leqslant d$.

For the special case when the coefficient b is independent of u in PDE (1) and independent of y in FBSDE (2), we have the following two results from Ouknine and Turpin [4]. Assume $b : [0, T] \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$, $\sigma : [0, T] \times \mathbb{R}^d \to \mathbb{R}^{d\times d}$, $g : [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ and $h : \mathbb{R}^d \to \mathbb{R}$ satisfy

(A1) b, σ are bounded continuous functions and there exists $K > 0$ such that

$$\left|b(t,x_1)-b(t,x_1)\right|+\left|\sigma(t,x_1)-\sigma(t,x_2)\right|\leqslant K|x_1-x_2|.$$

Moreover, $|(\sigma^*\nabla\sigma)(t,x_1)-(\sigma^*\nabla\sigma)(t,x_2)|\leqslant K|x_1-x_2|$.

(A2) There exists $K>0$ such that $|g(t,x,y_1,z_1)-g(t,x,y_2,z_2)|\leqslant K(|y_1-y_2|+|z_1-z_2|)$.

(A3) $\int_{\mathbb{R}^d}\int_0^T|g(t,x,0,0)|^2\,\mathrm{d}t\,\rho(x)\,\mathrm{d}x<\infty$, $\int_{\mathbb{R}^d}|h(x)|^2\rho(x)\,\mathrm{d}x<\infty$.

Proposition 2 *(Proposition 3.1 in [4]). Assume* (A1). *Then there exist two positive constants $\bar{k}$, $\bar{K}$ such that for any $t\leqslant s\leqslant T$ and $\phi\in L^1(\mathbb{R}^d,\rho(x)\,\mathrm{d}x)$, $\bar{k}\int_{\mathbb{R}^d}|\phi(x)|\rho(x)\,\mathrm{d}x\leqslant\int_{\mathbb{R}^d}E(|\phi(X_s^{t,x})|)\rho(x)\,\mathrm{d}x\leqslant\bar{K}\int_{\mathbb{R}^d}|\phi(x)|\rho(x)\,\mathrm{d}x$.*

Moreover, for any $\Phi\in L^1((0,T)\times\mathbb{R}^d,\mathrm{d}t\otimes\rho(x)\,\mathrm{d}x)$, $\bar{k}\int_{\mathbb{R}^d}\int_t^T|\Phi(s,x)|\,\mathrm{d}s\rho(x)\,\mathrm{d}x\leqslant\int_{\mathbb{R}^d}\int_t^T E(|\Phi(s,X_s^{t,x})|)\,\mathrm{d}s\,\rho(x)\,\mathrm{d}x\leqslant\bar{K}\int_{\mathbb{R}^d}\int_t^T|\Phi(s,x)|\,\mathrm{d}s\,\rho(x)\,\mathrm{d}x$.

From Lemma 4.1 and Theorem 4.8 in [4], we have

Proposition 3. *Assume* (A1)–(A3). *Then PDE* (1) *admits a unique Sobolev weak solution u. Moreover,*

$$u\big(s,X_s^{t,x}\big)=Y_s^{t,x},\quad (\sigma^*\nabla u)\big(s,X_s^{t,x}\big)=Z_s^{t,x},\quad \text{a.s., a.e. } s\in[t,T],\ x\in\mathbb{R}^d,$$

where $(X^{t,x},Y^{t,x},Z^{t,x})$ is the unique solution of FBSDE (2).

3. Sobolev weak solutions for PDEs in which *b* depends on *u*

In this section we consider the Sobolev weak solution of PDE (1) and relate it to the solution of FBSDE (2). Suppose $b:[0,T]\times\mathbb{R}^d\times\mathbb{R}\to\mathbb{R}^d$, $\sigma:[0,T]\times\mathbb{R}^d\to\mathbb{R}^{d\times d}$, $g:[0,T]\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}$ and $h:\mathbb{R}^d\to\mathbb{R}$ satisfy

(H1) b, σ are bounded continuous functions and there exists $K>0$ such that

$$\left|b(t,x_1,y_1)-b(t,x_1,y_2)\right|\leqslant K\big(|x_1-x_2|+|y_1-y_2|\big);$$
$$\left|\sigma(t,x_1)-\sigma(t,x_2)\right|+\left|(\sigma^*\nabla\sigma)(t,x_1)-(\sigma^*\nabla\sigma)(t,x_2)\right|\leqslant K|x_1-x_2|.$$

(H2) g is continuous in y and there exist $K>0$, $\mu\in\mathbb{R}$ such that

$$\left|g(t,x_1,y,z_1)-g(t,x_2,y,z_2)\right|\leqslant K\big(|x_1-x_2|+|z_1-z_2|\big);$$
$$(y_1-y_2)\big(g(t,x,y_1,z)-g(t,x,y_2,z)\big)\leqslant\mu|y_1-y_2|^2;$$
$$\left|g(t,x,y,z)\right|\leqslant\left|g(t,0,0,0)\right|+K\big(1+|x|+|y|+|z|\big).$$

(H3) $\int_0^T|g(t,0,0,0)|^2\,\mathrm{d}t<\infty$.

(H4) There exists $K>0$ such that $|h(x_1)-h(x_2)|\leqslant K|x_1-x_2|$.

Following Antonelli [1], we can easily obtain

Proposition 4. *Assume* (H1)–(H4). *Then there exists $T_0>0$ which depends on K, μ, such that when $T\leqslant T_0$, FBSDE* (2) *admits a unique solution $(X^{t,x},Y^{t,x},Z^{t,x})\in S^2(t,T;\mathbb{R}^d)\times S^2(t,T;\mathbb{R})\times M^2(t,T;\mathbb{R}^d)$ on $[t,T]$. Moreover,*

(i) *there exists $C_1>0$ which depends on K, μ such that $E\sup_{t\leqslant s\leqslant T}|X_s^{t,x}|^2+E\sup_{t\leqslant s\leqslant T}|Y_s^{t,x}|^2+E\int_t^T|Z_s^{t,x}|^2\,\mathrm{d}s\leqslant C_1(1+|x|^2+|h(0)|^2+\int_t^T(|b(s,0,0)|^2+|\sigma(s,0)|^2+|g(s,0,0,0)|^2)\,\mathrm{d}s)$;*

(ii) *there exists $C_2>0$ which depends on K, μ such that $E\sup_{t\leqslant s\leqslant T}|X_s^{t,x}-X_s^{t,x'}|^2+E\sup_{t\leqslant s\leqslant T}|Y_s^{t,x}-Y_s^{t,x'}|^2+E\int_t^T|Z_s^{t,x}-Z_s^{t,x'}|^2\,\mathrm{d}s\leqslant C_2^2|x-x'|^2$.*

We are now in a position to state our main result:

Theorem 5. *Assume* (H1)–(H4). *Then*

(i) *PDE* (1) *admits a local Sobolev weak solution u such that*

$$u\big(s, X_s^{t,x}\big) = Y_s^{t,x}, \quad (\sigma^*\nabla u)\big(s, X_s^{t,x}\big) = Z_s^{t,x}, \quad \textit{a.s., a.e. } s \in [t,T],\ x \in \mathbb{R}^d, \tag{3}$$

where $(X^{t,x}, Y^{t,x}, Z^{t,x})$ *is the unique solution of FBSDE* (2).

(ii) *If u is the Sobolev weak solution of PDE* (1) *and is assumed to be Lipschitz continuous in x, then u is unique.*

Proof. We first prove (i). By Proposition 4, there exists $T_0 = T_0(K,\mu) > 0$ such that when $T \leqslant T_0$, FBSDE (2) admits a unique solution $(X^{t,x}, Y^{t,x}, Z^{t,x})$ on $[t,T]$. If we define $u(t,x) = Y_t^{t,x}$, $v(t,x) = Z_t^{t,x}$ for $(t,x) \in [0,T] \times \mathbb{R}^n$, then u, v are deterministic functions and $|u(t,x) - u(t,x')| \leqslant C_2|x-x'|$. Moreover, from the uniqueness of the solutions to FBSDE (2), using the similar technique in Wu and Yu [8], we obtain

$$u\big(s, X_s^{t,x}\big) = Y_s^{s,X_s^{t,x}} = Y_s^{t,x}, \quad v\big(s, X_s^{t,x}\big) = Z_s^{s,X_s^{t,x}} = Z_s^{t,x} \quad \text{a.s., a.e. } s \in [t,T]. \tag{4}$$

Set $\bar b(s,x) = b(s,x,u(s,x))$, $\bar g(s,x) = g(s,x,u(s,x),v(s,x))$. Then $(X^{t,x}, Y^{t,x}, Z^{t,x})$ solves

$$\begin{cases} X_s^{t,x} = x + \int_t^s \bar b(r, X_r^{t,x})\,\mathrm{d}r + \int_t^s \sigma(r, X_r^{t,x})\,\mathrm{d}W_r, \\ Y_s^{t,x} = h\big(X_T^{t,x}\big) + \int_s^T \bar g\big(r, X_r^{t,x}\big)\,\mathrm{d}r - \int_s^T Z_r^{t,x} \cdot \mathrm{d}W_r. \end{cases} \tag{5}$$

It's easy to check that $|\bar b(t,x) - \bar b(t,x')| \leqslant (1+C_2)K|x-x'|$. Hence, $\bar b$ is Lipschitz in x. By Propositions 2, 4 and (4), there exists $C_3 = C_3(K,\mu,\bar k,\bar K,T) > 0$ such that $\int_t^T \int_{\mathbb{R}^d} (|u(s,x)|^2 + |v(s,x)|^2)\rho(x)\,\mathrm{d}x\,\mathrm{d}s \leqslant C_3 \int_{\mathbb{R}^d} (1 + |h(0)|^2 + |x|^2)\rho(x)\,\mathrm{d}x + C_3 \int_t^T (|b(s,0,0)|^2 + |\sigma(s,0)|^2 + |g(s,0,0,0)|^2)\,\mathrm{d}s \int_{\mathbb{R}^d} \rho(x)\,\mathrm{d}x < \infty$.

Consequently, it follows from (H2) and the definition of $\bar g$ that $\int_t^T \int_{\mathbb{R}^d} |\bar g(s,x)|^2 \rho(x)\,\mathrm{d}x\,\mathrm{d}s \leqslant 5K^2 \int_t^T \int_{\mathbb{R}^d} (|u(s,x)|^2 + |v(s,x)|^2)\rho(x)\,\mathrm{d}x\,\mathrm{d}s + 5\int_t^T |g(s,0,0,0)|^2\,\mathrm{d}s \int_{\mathbb{R}^d} \rho(x)\,\mathrm{d}x + 5K^2T \int_{\mathbb{R}^d} (1+|x|^2)\rho(x)\,\mathrm{d}x < \infty$. So the functions $\bar b$, σ, $\bar g$, h satisfy (A1)–(A3). Hence, Proposition 3 follows that $v = \sigma^*\nabla u$ and u is the Sobolev weak solution of the following PDE

$$\begin{cases} \frac{\partial u}{\partial t} + \sum_{i=1}^d \bar b_i(t,x) \frac{\partial u}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^d (\sigma\sigma^*)_{i,j}(t,x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \bar g(t,x) = 0, \\ u(T,x) = h(x). \end{cases} \tag{6}$$

Noting the definitions of $\bar b$, $\bar g$ and the fact that $v = \sigma^*\nabla u$, we can easily conclude that u is the Sobolev weak solution of PDE (1). And (3) is the immediate consequence of (4).

We now prove (ii). Suppose u is the Sobolev weak solution of PDE (1) and is Lipschitz continuous in x. If we set $\bar b(s,x) = b(s,x,u(s,x))$, $\bar g(s,x) = g(s,x,u(s,x),(\sigma^*\nabla u)(s,x))$, then $\bar b$ is Lipschitz continuous in x and $\int_{\mathbb{R}^d} \int_t^T |\bar g(s,x)|^2\,\mathrm{d}s\,\rho(x)\,\mathrm{d}x < \infty$. With the similar idea used in the proof of the uniqueness in Theorem 4.8 in [4], it easily follows that $\{X_s^{t,x}, u(s, X_s^{t,x}), (\sigma^*\nabla u)(s, X_s^{t,x}), t \leqslant s \leqslant T\}$ solves FBSDE (5). Hence, the uniqueness of the solutions of PDE (1) is a consequence of the uniqueness of the solutions of FBSDE (2). □

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