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Differential Geometry

The geometric complex for algebraic curves with cone-like singularities and admissible Morse functions

Ursula Ludwig

Mathematisches Institut, Eckerstrasse 1, 79104 Freiburg, Germany

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Abstract

In a previous Note the author gave a generalisation of Witten's proof of the Morse inequalities to the model of a singular complex algebraic curve X and a stratified Morse function f. In this Note a geometric interpretation of the complex of eigenforms of the Witten Laplacian corresponding to small eigenvalues is provided in terms of an appropriate subcomplex of the complex of unstable cells of critical points of f. ***To cite this article: U. Ludwig, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Un complexe géométrique sur des courbes algébriques complexes à singularités coniques et fonctions de Morse admissibles. Dans une Note précédente, l'auteur a donné une généralisation de la preuve de Witten des inegalités de Morse pour le cas modèle d'une courbe algébrique complexe singulière et d'une fonction de Morse stratifiée. Le but de cette Note est de donner une interprétation géométrique du complexe des formes propres du Laplacien de Witten pour de petites valeurs propres à l'aide d'un sous-complexe approprié du complexe des cellules instables. ***Pour citer cet article : U. Ludwig, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

1. Introduction

Let X be a singular complex algebraic curve. For simplicity of presentation we will assume that all singularities $\Sigma := \{p_1, \dots, p_N\}$, of X are unibranched. We denote by $m(p_i) = m_i \in \mathbb{N}$, $m_i \geqslant 2$, $i = 1, \dots, N$, the multiplicity of X at p_i. Let g be a metric on X such for each $p_i \in \Sigma$ there exists an open neighbourhood $U(p_i)$ in X such that $(U(p_i) - p_i, g_{|U_i - p_i})$ is isometric to $(\mathrm{cone}_\epsilon(S^1_{m_i}), \mathrm{d}r^2 + r^2\,\mathrm{d}\varphi^2)$ for some $\epsilon > 0$. Hereby for $m \in \mathbb{N}$ we denote by S^1_m the circle of length $2\pi m$ and by $\mathrm{cone}_\epsilon(S^1_m) := \{(r, \varphi) \mid r \in (0, \epsilon),\ \varphi \in S^1_m\}$.

Definition 1. Let $f : X \to \mathbb{R}$ be a continuous function which is smooth outside the singularities of X. The function f is called an admissible Morse function if the restriction $f_{|X-\Sigma}$ is Morse (in the smooth sense) and moreover if for

E-mail address: ursula.ludwig@math.uni-freiburg.de.

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any singular point $p \in \Sigma$ there exist $a_p, b_p \in \mathbb{R}$, $(a_p, b_p) \neq (0,0)$, such that the function f has the following form in local coordinates (r, φ) near p: $f(r,\varphi) = f(p) + r(a_p \cos(\varphi) + b_p \sin(\varphi))$.

As explained in [5] (X, g) is a metric model for the singular complex algebraic curve $X \subset \mathbb{P}^n$, equipped with the metric induced by the Fubini–Study metric on $\mathbb{P}^n$. Admissible Morse functions are particular examples of stratified Morse functions in the sense of the theory developed in [2]. In [5] a generalisation of Witten's proof of the Morse inequalities is given for the singular situation defined above. For singular spaces with cone-like singularities the L^2-cohomology $H^*_{(2)}(X)$, i.e. the cohomology of the complex $(\mathcal{C}, d, \langle,\rangle)$ of L^2-integrable forms, is a topological invariant. Note that $H^*_{(2)}(X)$ can also be computed by considering only the subcomplex $(\mathcal{D}, d, \langle,\rangle)$ of smooth L^2-integrable forms on $X \setminus \Sigma$. The generalisation of Witten's method to the singular algebraic curve consists in deforming the complex $(\mathcal{C}, d, \langle,\rangle)$ by means of the admissible Morse function f into a complex $(\mathcal{C}_t, d_t, \langle,\rangle)$. We denote by Δ_t the Witten Laplacian associated to the deformed complex. The Morse inequalities for L^2-cohomology shown in [5] follow from the properties of the subcomplex $(\mathbb{F}_t, d_t, \langle,\rangle)$ generated by the eigenforms of Δ_t corresponding to the eigenvalues in $[0,1]$. In the smooth situation Witten [6] conjectured that for $t \to \infty$ the complex of eigenforms corresponding to small eigenvalues converges to the so-called Thom–Smale complex. A proof of this fact based on semi-classical analysis has been given by Helffer and Sjöstrand in [3]. A different proof relying on a result by Laudenbach [4] was given by Bismut and Zhang in [1]. The goal of this Note is to provide a geometric interpretation of the complex of $(\mathbb{F}_t, d_t)$ in terms of an appropriate subcomplex of the complex of unstable cells for the critical points of the admissible Morse function f. The proofs given here extend ideas in [1] to the singular situation.

2. The combinatorial complex $(C^{u'}_*, \partial_*)$

Let us denote by $\mathrm{Crit}(f)$ the set of critical points for f and by $\mathrm{Crit}_i(f)$, $i = 0,1,2$, the set of critical points of index i. Hereby we consider $\Sigma \subset \mathrm{Crit}_1(f)$. Let us denote by $-\nabla_g f$ the negative gradient vector field of f and by Φ the induced flow. Note that the flow Φ is not defined for all time $t \in \mathbb{R}$. However all critical points of f are fixed points of the flow, and we can still define the stable/unstable sets: $W^{u/s}(p) = \{x \in X \mid \text{there exist } t^-(x) < 0,\ t^+(x) > 0 \text{ such that } \lim_{t \to t^{\mp}(x)} \Phi(x,t) = p\}$. If $p \in X - \Sigma$ is a critical point for f of index $\mathrm{ind}(p)$ it is well known that the stable resp. unstable manifold is a (non closed) manifold of dimension $\dim W^s(p) = 2 - \mathrm{ind}(p)$ resp. $\dim W^u(p) = \mathrm{ind}(p)$. It is not difficult to see that for $p \in \Sigma$ the sets $W^{u/s}(p) - \{p\}$ are manifolds of dimension 1 having $m(p)$ connected components. By rescaling and changing the coordinate on the link one can always assume that $(a_p, b_p) = (1,0)$ in Definition 1. Then locally near p the $m(p)$ connected components of $W^u(p)$ are $W^u_j(p) = \{(r,\varphi) \in U(p) \mid r \in \mathbb{R}^+,\ \varphi = (2j+1)\pi\}$ for $j \in \widetilde{I}_p := \mathbb{Z}/m(p)\mathbb{Z}$. The following proposition describes the boundary of the unstable sets. It is a generalisation of Proposition 2 in [4] to our situation.

Proposition 2. *Let f be an admissible Morse function such that $\nabla_g f$ satisfies the Morse–Smale condition. Then for each critical point $p \in \mathrm{Crit}(f)$ the closure $\overline{W^u(p)}$ is a stratified space. Let $p \in \mathrm{Crit}(f) \setminus \Sigma$. Then the strata of $\overline{W^u(p)} \setminus W^u(p)$ can be of the following form*: (a) *$W^u(q)$, for $q \in \mathrm{Crit}(f) - \Sigma$, $\mathrm{ind}(q) < \mathrm{ind}(p)$.* (b) *$W^u_j(q)$, for $q \in \Sigma$, $j \in \widetilde{I}_q$ and $1 = \mathrm{ind}(q) < \mathrm{ind}(p)$,* (c) *$\{q\}$, where $q \in \Sigma$, $\mathrm{ind}(q) < \mathrm{ind}(p)$.*

Moreover the strata of type (b) *"come in pairs", i.e. one of the following possibilities holds*:

(1) *If there exists $j \in \widetilde{I}_q$ such that $W^u_j(q) \subset \partial W^u(p)$ then either $W^u_{j-1}(q) \subset \partial W^u(p)$ or $W^u_{j+1}(q) \subset \partial W^u(p)$.*
(2) *$W^u(p)$ has $2n$ connected components near $W^u_j(q)$ such that $W^u_j(q)$ is the boundary of n of these, while $-W^u_j(q)$ is the boundary of the other n.*

Moreover for $p \in \Sigma$, $j \in \widetilde{I}_p$ we have $W^u_j(p) \simeq (0,1)$ and $\overline{W^u_j(p)} \simeq [0,1]$ where one end of the compactification corresponds to p and the other end corresponds to some $q \in \mathrm{Crit}_0(f)$.

We choose an orientation on the W^u's, $W^u_j(p)$ is oriented by the flow for $p \in \Sigma$, $j \in \widetilde{I}_p$. The chain groups of the complex (C^u_*, ∂_*) are defined as follows: $C^u_2 := \bigoplus_{p \in \mathrm{Crit}_2(f)} \mathbb{R} \cdot [W^u(p)]$,

$$C^u_1 := \bigoplus_{p \in \mathrm{Crit}_1(f),\ p \notin \Sigma} \mathbb{R} \cdot \big[W^u(p)\big] \oplus \bigoplus_{p \in \Sigma,\ j \in \widetilde{I}_p} \mathbb{R} \cdot \big[W^u_j(p)\big], \qquad C^u_0 := \bigoplus_{p \in \mathrm{Crit}_0(f)} \mathbb{R} \cdot \big[W^u(p)\big] \oplus \bigoplus_{p \in \Sigma} \mathbb{R} \cdot \big[\{p\}\big].$$

Note that since $\mathrm{Crit}(f)$ is finite the above chain groups are well-defined. The boundary of a generator $\sigma \in C_i^u$ is defined by $\partial\sigma = \sum n(\sigma,\theta)\cdot\theta$, where the sum is taken over all generators of C_{i-1}^u and where $n(\sigma,\theta)=0$ if θ is not in the closure of σ. Moreover, if θ is in the closure of σ the coefficient $n(\sigma,\theta)$ is defined as follows: Near θ the cell σ has $n=n_+ + n_-$ connected components such that θ is the oriented boundary of n_+ of these and $-\theta$ is the oriented boundary of the other n_-. Then $n(\sigma,\theta)=n_+ - n_-$. Let us denote by $(C_*^{u'},\partial_*)$ the subcomplex of (C_*^u,∂_*) generated by the cells $\{[W^u(p)] \mid p\in \mathrm{Crit}(f)\setminus\Sigma\}$ and $\{\sigma_i^u(p) := [W_i^u(p)] - [W_{i-1}^u(p)],\ p\in\Sigma,\ i=1,\dots,m-1\}$. The fact that $(C_*^{u'},\partial_*)$ is indeed a subcomplex follows from Proposition 2.

Note that the complex $(C_*^{u'},\partial_*)$ computes the intersection homology $IH_*(X)$ of X: The decomposition of X into unstable cells is a CW-decomposition of X and therefore $H_*((C_*^u,\partial_*)) \simeq H_{\mathrm{sing}}(X)$. Since X is a unibranched curve and therefore (topologically) normal we have that $H_{\mathrm{sing}}(X)\simeq IH_*(X)$. Moreover the inclusion of complexes $(C_*^{u'},\partial_*) \hookrightarrow (C_*^u,\partial_*)$ is a quasi-isomorphism. Note however that the cells σ_j^u are not allowed in the sense of intersection homology.

Proposition 3. *Let $\beta\in\mathcal{D}^i$, i.e. both β and $d\beta$ are square integrable. Let $\sigma\in C_i^{u'}$. Then the integral $\int_\sigma\beta$ exists and Stokes' theorem holds. Thus integration yields a well-defined morphism of complexes $\int : (C_i^{u'},\partial)\times(\mathcal{D}^i,d)\to\mathbb{R}$.*

3. The complex of eigenforms to small eigenvalues $(\mathbb{F}_t, d_t)$

Denote by $(\mathbb{F}_t, d_t, \langle,\rangle)$ the subcomplex of $(\mathcal{C}_t, d_t,\langle,\rangle)$ generated by the eigenforms of Δ_t corresponding to eigenvalues $\lambda\in[0,1]$. We denote by $P(t,[0,1])$ the orthogonal projection operator from $\mathcal{C}_t$ on $\mathbb{F}_t$ (with respect to the metric $\langle,\rangle$). For $p\in\Sigma$ of multiplicity m we set $I_p := \{1,\dots,m-1\}$. For a smooth critical point let $I_p := \{1\}$. Let us shortly recall the construction of a basis for $(\mathbb{F}_t,d_t)$. As shown in [5] for a singular point $p\in\Sigma$ of multiplicity m we can construct a local model operator $\mathbf{\Delta}_t^p$ on the infinite cone with the property that $\mathrm{spec}(\mathbf{\Delta}_t^p)\subset\{0\}\cup[t^2,\infty)$ and $\dim\ker(\mathbf{\Delta}_t^p)=m-1$. We assume again that $(a_p,b_p)=(1,0)$ in Definition 1 and that m is odd. For $\nu\in\{\frac{1}{m},\dots,\frac{m-1}{2m}\}$ set

$$\gamma_\nu^1 := \mathrm{tr}\big(K_{\nu-1}(\mathrm{tr})\cos(\nu\varphi) + K_\nu(\mathrm{tr})\cos\big((\nu-1)\varphi\big)\big)\,\mathrm{d}\varphi + t\big(K_{\nu-1}(\mathrm{tr})\sin(\nu\varphi) - K_\nu(\mathrm{tr})\sin\big((\nu-1)\varphi\big)\big)\,\mathrm{d}r,$$
$$\gamma_\nu^2 := \mathrm{tr}\big(K_{\nu-1}(\mathrm{tr})\sin(\nu\varphi) + K_\nu(\mathrm{tr})\sin\big((\nu-1)\varphi\big)\big)\,\mathrm{d}\varphi + t\big(-K_{\nu-1}(\mathrm{tr})\cos(\nu\varphi) + K_\nu(\mathrm{tr})\cos\big((\nu-1)\varphi\big)\big)\,\mathrm{d}r,$$

where K_ν is the modified Bessel function of the second kind. For $j\in I_p$ we denote by $\nu_j := \frac{j}{m}$ if $j\leqslant\frac{m-1}{2}$ and $\nu_j := \frac{j}{m}-\frac{m-1}{2m}$ if $\frac{m}{2}<j\leqslant m-1$. Define $\omega_p^j := \gamma_{\nu_j}^1$ for $j\leqslant\frac{m-1}{2}$ and $\omega_p^j := \gamma_{\nu_j}^2$ for $\frac{m}{2}<j\leqslant m-1$. Then $\ker(\mathbf{\Delta}_t^p) = \mathrm{span}\{\omega_p^j(t),\ j\in I_p\}$. In case m is even one has to add the form $\omega_p^{m/2} := \mathrm{tr}\,K_{1/2}\cos(1/2\varphi)\,\mathrm{d}\varphi + tK_{1/2}\sin(1/2\varphi)\,\mathrm{d}r$. The model operator for a smooth critical point is well-known and in this case we have that $\ker(\mathbf{\Delta}_t^p)=\mathrm{span}\{\omega_p^1\}$. Let $\nu_\epsilon : \mathbb{R}^+\to\mathbb{R}$ be a cut-off function with $\nu_\epsilon = 1$ in $[0,\epsilon/2]$. The forms $\Phi_p^j(t) := \beta_p^j(t)^{-1}\nu_\epsilon(\|x\|)\omega_p^j(t)$, where $\beta_p^j(t) := \|\nu_\epsilon(\|x\|)\omega_p^j(t)\|$, can be identified with forms in $(\mathcal{C}_t,d_t,\langle,\rangle)$. Note that $\{P(t,[0,1])(\Phi_j^p(t)),\ p\in\mathrm{Crit}(f),\ j\in I_p\}$ yields a basis of $\mathbb{F}_t$ which is orthonormal up to a term of order $\mathrm{O}(e^{-ct})$.

4. Comparison theorem

Lemma 4. *Let $p\in\Sigma$. There exists $A := (a_{li})\in\mathrm{GL}_{m-1}(\mathbb{R})$ such that $\int_{\tilde e_p^i}\omega_p^j = \delta_{ij}$, where $\tilde e_p^i := \sum_{l=1}^{m-1} a_{li}([L_l]-[L_{l-1}])$, $L_l := \{(r,\varphi)\in\mathrm{cone}(S_m^1)\mid\varphi=(2l+1)\pi\}$.*

For $p\in\mathrm{Crit}(f)\setminus\Sigma$ define $e_p^1 := [W^u(p)]$, $W^u(e_p^1) := W^u(p)$. For $p\in\Sigma$ and $j\in I_p$ define

$$e_p^j := \sum_{l\in I_p} a_{lj}\big[\sigma_l^u(p)\big], \qquad W^u\big(e_p^j\big) := \sum_{l\in I_p} a_{lj}\sigma_l^u(p).$$

We denote by $\widetilde\Delta_t$ the Laplacian associated to the complex $(\mathcal{C},d,\langle,\rangle_t)$, where $\langle,\rangle_t$ is the twisted metric $\langle\alpha,\beta\rangle_t := \int\alpha\wedge*\beta e^{-2ft}$. Denote by $(\widetilde{\mathbb{F}}_t,d,\langle,\rangle_t)$ the subcomplex of eigenforms corresponding to eigenvalues of $\widetilde\Delta_t$ in $[0,1]$. Obviously the map $\omega\mapsto e^{ft}\omega$ induces an isomorphism of complexes $(\mathbb{F}_t,d_t,\langle,\rangle)\to(\widetilde{\mathbb{F}}_t,d,\langle,\rangle_t)$. Denote by $R_i(t)$

the linear map $R_i(t) : \mathrm{Hom}(C_i^{u'}, \mathbb{R}) \to \widetilde{\mathbb{F}}_t^i$, $[e_p^j]^* \mapsto e^{ft} P(t, [0,1])(\widetilde{\Phi}_p^j)$. For t large enough $R_i(t)$ is an isomorphism of vector spaces. Note that by elliptic regularity the complex $(\widetilde{\mathbb{F}}_t, d)$ can be considered as a subcomplex of the complex $(\mathcal{D}, d)$ and therefore by Proposition 3 the integration morphism $P_{\infty,t} : (\widetilde{\mathbb{F}}_t^i, d) \to \mathrm{Hom}(C_i^{u'}, \mathbb{R})$, $\omega \mapsto \sum_{p \in \mathrm{Crit}_i(f), j \in I_p} (\int_{\overline{W^u(e_p^j)}} \omega)[e_p^j]^*$ is well-defined. By Stokes' theorem $P_{\infty,t}$ is a morphism of complexes. Denote by $\mathcal{F} \in \mathrm{End}(\mathrm{Hom}(C_i^{u'}, \mathbb{R}))$ the linear map that acts on $[e_p^j]^*$ by multiplication with $f(p)$. Let $\mathcal{I} \in \mathrm{End}(\mathrm{Hom}(C_i^{u'}, \mathbb{R}))$ denote the multiplication by i. One can now prove the following two results, which generalise Theorem 6.11 and Theorem 6.12 in [1] respectively to the singular situation.

Theorem 5. (a) *The asymptotic behaviour of $P_{\infty,t} \circ R(t)$ as $t \to \infty$ is $P_{\infty,t} \circ R(t) = e^{t\mathcal{F}} (\frac{\pi}{t})^{(\mathcal{I}-1)/2} (1 + \mathrm{O}(e^{-ct}))$. In particular for large t the homomorphism of vector spaces $P_{\infty,t}$ is an isomorphism.* (b) *There exists $c > 0$ such that for $t \to \infty$, $R(t)^{-1} \circ d \circ R(t) = \sqrt{\frac{t}{\pi}} (1 + \mathrm{O}(e^{-ct}))^{-1} e^{-t\mathcal{F}} \partial^* e^{t\mathcal{F}} (1 + \mathrm{O}(e^{-ct}))$.*

The proof of Theorem 5(a) is similar to that of Theorem 6.11 in [1]. In the singular situation one uses Proposition 2, Lemma 4 as well as Proposition 7 in [5]. Part (b) follows directly from (a). The theorem above shows that for $t \to \infty$ the complex of eigenforms corresponding to small eigenvalues converges to the geometric complex $(C_*^{u'}, \partial_*)$. Thus, since we know that $H^*((\mathbb{F}_t, d_t, \langle , \rangle)) \simeq H_{(2)}^*(X)$ Theorem 5 gives a second proof of the fact, that the complex $(C_*^{u'}, \partial_*)$ computes the intersection homology of X.

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