Available online at www.sciencedirect.com
ScienceDirect
C. R. Acad. Sci. Paris, Ser. I 347 (2009) 791–796
COMPTES RENDUS MATHEMATIQUE

Differential Geometry

Multiplicative noncommutative deformations of left invariant Riemannian metrics on Heisenberg groups

Amine Bahayou [a], Mohamed Boucetta [b]

[a] *Université Ouargla, BP 511, route Ghardaïa, 30000 Ouargla, Algeria*
[b] *Faculté des sciences et techniques Gueliz, BP 549, 40000 Marrakech, Morocco*

Received 29 December 2008; accepted 17 April 2009

Available online 9 May 2009

Presented by Charles-Michel Marle

Abstract

Let H_n be the Heisenberg group of dimension $2n+1$. We give a precise description of all $(\pi, \langle,\rangle)$, where π is a multiplicative Poisson tensor on H_n and $\langle,\rangle$ is a left invariant metric on H_n such that $(\pi, \langle,\rangle)$ satisfies the necessary conditions, introduced by Eli Hawkins, to the existence of a noncommutative deformation of the spectral triple associated to $(H_n, \langle,\rangle)$. ***To cite this article: A. Bahayou, M. Boucetta, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Déformations non commutatives et multiplicatives des métriques invariantes à gauche sur les groupes de Heisenberg. Soit H_n le groupe de Heisenberg de dimension $2n+1$. Nous donnons une description complète des couples $(\pi, \langle,\rangle)$ où π est un tenseur de Poisson multiplicatif et $\langle,\rangle$ une métrique invariante à gauche sur H_n tels que $(\pi, \langle,\rangle)$ vérifie les conditions nécessaires, introduites par Eli Hawkins, à l'existence d'une déformation non commutative du triple spectral associé à $(H_n, \langle,\rangle)$. ***Pour citer cet article : A. Bahayou, M. Boucetta, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Version française abrégée

Dans [4] et [5], Hawkins a montré que si une déformation de l'algèbre des formes différentielles d'une variété riemannienne (P, g) provient d'une déformation du triple spectral décrivant la structure riemannienne, alors le tenseur de Poisson π (associé à la déformation) et la métrique riemannienne vérifient les conditions suivantes :

(i) la connexion de Levi-Civita contravariante $\mathcal{D}$ associée au couple (π, g) est plate,
(ii) la métacourbure de $\mathcal{D}$ est nulle,
(iii) le tenseur de Poisson est compatible avec le volume riemannienne μ, c'est-à-dire, $d(i_\pi \mu) = 0$.

E-mail addresses: amine_bahayou@yahoo.fr (A. Bahayou), mboucetta2@yahoo.fr (M. Boucetta).

1631-073X/$ – see front matter
doi:10.1016/j.crma.2009.04.013

La connexion de Levi-Civita contravariante $\mathcal{D}$ associée au couple (π, g) est l'analogue de la connexion de Levi-Civita classique ; elle a été introduite dans [1]. La métacourbure est un $(2,3)$ champ de tensor, elle a été introduite par Hawkins dans [5]. Un couple (π, g) satisfaisant (i)–(iii) sera dit compatible dans le sens de Hawkins.

Dans cette Note, nous allons démontrer les deux théorèmes suivants :

Théorème 0.1. *Soit* $(G, \pi, \langle,\rangle)$ *un groupe de Lie unimodulaire et connexe muni d'un tenseur de Poisson multiplicatif et d'une métrique riemannienne invariante à gauche. Notons* ξ *le* 1*-cocycle associé à* π *et* μ *le volume riemannien. Alors* $d(i_\pi \mu) = 0$ *si et seulement si l'algèbre de Lie dual* $\mathcal{G}^*$ *est unimodulaire et, pour tout* $u \in \mathcal{G}$, $\delta(i_{\xi(u)}\mu_e) = 0$, *où* δ *est de la différentielle de* $\wedge^*\mathcal{G}^*$.

Soit H_n le groupe de Heisenberg de dimension $2n+1$, $\mathcal{H}_n$ son algèbre de Lie et z un élément non-nul du centre de $\mathcal{H}_n$. Il existe une 2-forme ω sur $\mathcal{H}_n$, telle que $i_z\omega = 0$, la projection de ω sur $\mathcal{H}_n/(\mathbb{R}z)$ est non-dégénérée et, pour tous $u, v \in \mathcal{H}_n$, $[u, v] = \omega(u, v)z$.

Théorème 0.2. *Soient* π *et* $\langle,\rangle$, *respectivement, un tenseur de Poisson multiplicatif et une métrique riemannienne invariante à gauche sur* H_n. *Alors* π *et* $\langle,\rangle$ *sont compatibles dans le sens de Hawkins si et seulement si* :

(i) *il existe un endomorphisme* $J : \mathcal{H}_n \to \mathcal{H}_n$ *anti-symétrique par rapport à* $\langle,\rangle_e$ *tel que* $J(z) = 0$ *et, pour tout* $u \in \mathcal{H}_n$, $\xi(u) = z \wedge Ju$,
(ii) *pour tous* $u, v \in \mathcal{H}_n$, $\omega(J^2u, v) + \omega(u, J^2v) + 2\omega(Ju, Jv) = 0$.

Noter que dans [8] tous les tenseurs de Poisson multiplicatifs sur H_n ont été déterminés, et dans ce théorème, nous retrouvons une famille de ces tenseurs.

1. Introduction and main results

In [4] and [5], Hawkins showed that if a deformation of the graded algebra of differential forms on a Riemannian manifold (P, g) comes from a deformation of the spectral triple describing the Riemannian manifold P, then the Poisson tensor π (which characterizes the deformation) and the Riemannian metric satisfy the following conditions:

(i) the Levi-Civita contravariant connection $\mathcal{D}$ associated to (π, g) is flat,
(ii) the metacurvature of $\mathcal{D}$ vanishes,
(iii) the Poisson tensor π is compatible with the Riemannian volume μ, i.e., $d(i_\pi \mu) = 0$.

The Levi-Civita contravariant connection associated naturally to any couple of a pseudo-Riemannian metric and a Poisson tensor is an analogue of the Levi-Civita connection. It has appeared first in [1]. The metacurvature of a torsion-free and flat contravariant connection, introduced by Hawkins in [5], is a $(2,3)$-tensor field. We call a pseudo-Riemannian metric and a Poisson tensor satisfying the conditions (i)–(iii) *compatible in the sense of Hawkins*.

In this Note, we give a complete description of $(H_n, \pi, \langle,\rangle)$, where H_n is the Heisenberg group of dimension $2n+1$ endowed with a multiplicative Poisson tensor π and a left invariant Riemannian metric $\langle,\rangle$ which are compatible in the sense of Hawkins. Before stating our main results, let us recall briefly some classical facts about multiplicative Poisson tensors and Heisenberg groups. Many fundamentals definitions and results about Poisson manifolds and multiplicative Poisson tensors can be found in [2].

A Poisson tensor π on a Lie group G is called multiplicative if, for any $a, b \in G$, $\pi(ab) = (L_a)_*\pi(b) + (R_b)_*\pi(a)$. Pulling π back to the identity element e of G by left translations, we get a map $\pi_l : G \to \mathcal{G} \wedge \mathcal{G}$ defined by $\pi_l(g) = (L_{g^{-1}})_*\pi(g)$. Let $\xi := d_e\pi_l : \mathcal{G} \to \mathcal{G} \times \mathcal{G}$ be the intrinsic derivative of π_l at e. It is well known that ξ is a 1-cocycle relative to the adjoint representation of $\mathcal{G}$ on $\mathcal{G} \wedge \mathcal{G}$, and, the dual map of ξ, $[,]^* : \mathcal{G}^* \times \mathcal{G}^* \to \mathcal{G}^*$, is exactly the Lie bracket on $\mathcal{G}^*$ obtained by linearizing the Poisson structure at the unit element e. Note that if one identify $\mathcal{G}^*$ to the space of left invariant 1-forms, then $[,]^*$ is identified with Koszul bracket.

The Heisenberg group H_n of dimension $2n+1$ is the group of matrices

$$\begin{pmatrix} 1 & X & c \\ 0 & I_n & {}^tY \\ 0 & 0 & 1 \end{pmatrix},$$

where $c \in \mathbb{R}$ and $X, Y \in \mathbb{R}^n$. Let z be a non-nul central element in the Lie algebra $\mathcal{H}_n$ of H_n. Then there exists a 2-form ω on $\mathcal{H}_n$ such that $i_z\omega = 0$, the projection of ω on $\mathcal{H}_n/\mathbb{R}z$ is non-degenerate and, for any $u, v \in \mathcal{H}_n$, $[u, v] = \omega(u, v)z$. Let us state now our main results:

Theorem 1.1. *Let π and $\langle,\rangle$ be, respectively, a multiplicative Poisson tensor and a left invariant Riemannian metric on a connected and unimodular Lie group G, and let μ be the Riemannian volume form associated to $\langle,\rangle$. Then $di_\pi\mu = 0$ if and only if $\mathcal{G}^*$ is an unimodular Lie algebra and, for any $v \in \mathcal{G}$, $\delta(i_{\xi(v)}\mu_e) = 0$, where $\delta : \wedge^*\mathcal{G}^* \to \wedge^*\mathcal{G}^*$ is the differential.*

Theorem 1.2. *Let π and $\langle,\rangle$ be, respectively, a multiplicative Poisson tensor and a left invariant Riemannian metric on the Heisenberg group H_n. Then $(\pi, \langle,\rangle)$ are compatible in the sense of Hawkins if and only if*:

(i) *there exists an endomorphism $J : \mathcal{H}_n \to \mathcal{H}_n$ skew-symmetric with respect to $\langle,\rangle_e$ such that $J(z) = 0$ and, for any $u \in \mathcal{H}_n$, $\xi(u) = z \wedge Ju$,*
(ii) *for any $u, v \in \mathcal{H}_n$, $\omega(J^2u, v) + \omega(u, J^2v) + 2\omega(Ju, Jv) = 0$.*

In [8], all multiplicative Poisson tensors on H_n are described, and we recover in Theorem 1.2 a class of such tensors.

The Note is organized as follows: In Section 2, we recall some basic fact about Levi-Civita contravariant connections and about the notion of metacurvature. We give also an improved version of a result of Milnor (see [7], Theorem 1.5) which will play a crucial role in the proof of Theorem 1.2. In Section 3, we prove Theorems 1.1 and 1.2 and we give some examples.

2. Contravariant connections and metacurvature

Many fundamental definitions and results about contravariant connections can be found in [3].

For any Poisson manifold (P, π), we denote by $\pi_\# : T^*P \to TP$ the anchor map given by $\beta(\pi_\#(\alpha)) = \pi(\alpha, \beta)$, and by $[,]_\pi$ the Koszul bracket on $\Omega^1(P)$ given by $[\alpha, \beta]_\pi = L_{\pi_\#(\alpha)}\beta - L_{\pi_\#(\beta)}\alpha - d(\pi(\alpha, \beta))$. Note that the Koszul bracket can be extended to all $\Omega^*(P)$ and will be denoted in the same manner.

Let $(P, \pi, \langle,\rangle)$ be a Poisson manifold endowed with a pseudo-Riemannian metric. Denote by $\langle,\rangle^*$ the pseudo-Euclidian structure induced by $\langle,\rangle$ on T^*P. The Levi-Civita contravariant connection associated to $(\pi, \langle,\rangle)$ is the unique contravariant connection $\mathcal{D}$ such that $\mathcal{D}$ is torsion-free, i.e., $\mathcal{D}_\alpha\beta - \mathcal{D}_\beta\alpha = [\alpha, \beta]_\pi$, and the metric $\langle,\rangle$ is parallel with respect to $\mathcal{D}$, i.e., $\pi_\#(\alpha).\langle\beta, \gamma\rangle^* = \langle\mathcal{D}_\alpha\beta, \gamma\rangle^* + \langle\beta, \mathcal{D}_\alpha\gamma\rangle^*$. The connection $\mathcal{D}$ is the contravariant analogue of the Levi-Civita connection and can be defined by the Koszul formula:

$$\begin{aligned} 2\langle\mathcal{D}_\alpha\beta, \gamma\rangle^* = {}& \pi_\#(\alpha).\langle\beta, \gamma\rangle^* + \pi_\#(\beta).\langle\alpha, \gamma\rangle^* - \pi_\#(\gamma).\langle\alpha, \beta\rangle^* \\ & + \langle[\gamma, \alpha]_\pi, \beta\rangle^* + \langle[\gamma, \beta]_\pi, \alpha\rangle^* + \langle[\alpha, \beta]_\pi, \gamma\rangle^*. \end{aligned} \tag{1}$$

The curvature of $\mathcal{D}$ is formally identical to the usual definition $K(\alpha, \beta) = \mathcal{D}_\alpha\mathcal{D}_\beta - \mathcal{D}_\beta\mathcal{D}_\alpha - \mathcal{D}_{[\alpha,\beta]_\pi}$. The connection $\mathcal{D}$ is called flat if K vanishes identically.

In [5], Hawkins showed that such a connection defines a bracket $\{,\}$ on the space of differential forms $\Omega^*(P)$ such that:

(i) $\{\sigma, \rho\} = -(-1)^{\deg\sigma \deg\rho}\{\rho, \sigma\}$.
(ii) $d\{\sigma, \rho\} = \{d\sigma, \rho\} + (-1)^{\deg\sigma}\{\sigma, d\rho\}$.
(iii) $\{\sigma, \rho \wedge \lambda\} = \{\sigma, \rho\} \wedge \lambda + (-1)^{\deg\sigma \deg\rho}\rho \wedge \{\sigma, \lambda\}$.
(iv) For any $f, h \in C^\infty(P)$ and for any $\sigma \in \Omega^*(P)$ the bracket $\{f, h\}$ coincides with the initial Poisson bracket and $\{f, \sigma\} = \mathcal{D}_{df}\sigma$.

Hawkins called this bracket *generalized Poisson bracket*. The following formula gives the generalized Poisson bracket of $\alpha, \beta \in \Omega^1(P)$ and can be stated easily:

$$\{\alpha, \beta\} = -\mathcal{D}_\alpha d\beta - \mathcal{D}_\beta d\alpha + d\mathcal{D}_\beta\alpha + [\alpha, d\beta]_\pi. \tag{2}$$

If $\mathcal{D}$ is flat, Hawkins showed that there exists a $(2,3)$-tensor $\mathcal{M}$ such that the generalized Poisson bracket satisfies the graded Jacobi identity if and only if $\mathcal{M}$ vanishes identically. $\mathcal{M}$ is called the *metacurvature* of $\mathcal{D}$ and is given by

$$\mathcal{M}(df, \alpha, \beta) = \big\{f, \{\alpha, \beta\}\big\} - \big\{\{f, \alpha\}, \beta\big\} - \big\{\{f, \beta\}, \alpha\big\}. \tag{3}$$

Moreover, Hawkins pointed out in [5], p. 394, that for any parallel 1-forms α, β and for any 1-form γ,

$$\mathcal{M}(\alpha, \beta, \gamma) = -\mathcal{D}_\gamma\mathcal{D}_\beta d\alpha. \tag{4}$$

Finally, let us recall a result of Milnor which will play a crucial role in the proof of Theorem 1.2. Let $(G, \langle,\rangle)$ be a Lie group endowed with a left invariant Riemannian metric, $\mathcal{G}$ the Lie algebra of G and $S_{\langle,\rangle} = \{u \in \mathcal{G}, ad_u + ad_u^t = 0\}$ where $ad_u(v) = [u, v]$ and ad_u^t is the adjoint of ad_u with respect to $\langle,\rangle_e$. Note that $S_{\langle,\rangle}$ is a subalgebra of $\mathcal{G}$. The following result is an improvement of a classical result of Milnor (see [7], Theorem 1.5), we omit its proof:

Theorem 2.1. *The curvature of $\langle,\rangle$ vanishes identically if and only if $S_{\langle,\rangle}$ is abelian, the derived ideal $[\mathcal{G}, \mathcal{G}]$ is abelian of even dimension $2r$ and its orthogonal is $S_{\langle,\rangle}$. In this case, there exists $(u_1, \dots, u_r)$ a family of vectors in $S_{\langle,\rangle} \setminus \{0\}$ and an orthonormal basis $(e_1, f_1, \dots, e_r, f_r)$ of $[\mathcal{G}, \mathcal{G}]$ such that, for any $s \in S_{\langle,\rangle}$ and for $i = 1, \dots, r$, $[s, e_i] = \langle s, u_i\rangle f_i$ and $[s, f_i] = -\langle s, u_i\rangle e_i$.*

3. Sketch of the proof of Theorems 1.1 and 1.2

3.1. Sketch of the proof of Theorem 1.1

We consider the modular vector field X_μ given by $di_\pi\mu = i_{X_\mu}\mu$ and the modular form of $\mathcal{G}^*$, $\kappa : \mathcal{G}^* \to \mathbb{R}$, given by $\kappa(\alpha) = \mathrm{Tr}\, ad_\alpha$, where $ad_\alpha\beta = [\alpha, \beta]^*$. The modular form defines a vector in $\mathcal{G}$, we denote it also by κ. Denote by κ^+ the left invariant vector field associated to κ.

The following relation between the modular vector field and the modular form can be stated by a direct computation using $di_\pi\mu = i_{X_\mu}\mu$, an orthonormal frame of left invariant vector fields and $\pi(e) = 0$,

$$X_\mu(e) = \kappa. \tag{5}$$

Consider the following formula, due to Koszul (see [6]),

$$i_{[X,Q]}\mu = i_X di_Q\mu + (-1)^{\deg Q} di_X i_Q\mu - i_Q di_X\mu \tag{6}$$

which holds for any vector field X and any multi-vector field Q. Let X be a left invariant vector field. Since G is unimodular, we have $L_X\mu = 0$. By using this fact and by applying (6) to $[X, X_\mu]$ and $[X, \pi]$, we get

$$d(i_{[X,\pi]}\mu) = -i_{[X,X_\mu]}\mu. \tag{7}$$

Since $[X, \pi]$ and μ are left invariant, we deduce that $[X, X_\mu]$ is left invariant. From (5), we deduce that $X_\mu = X_m + \kappa^+$ where X_m is a multiplicative vector field. Hence X_μ vanishes if and only if $X_m = \kappa = 0$. Or X_m is multiplicative and it vanishes if and only if $[X, X_m](e) = 0$ for any left invariant vector field X and the theorem follows from (7). □

3.2. Sketch of the proof of Theorem 1.2

We organize the proof in three steps:

(i) *The vanishing of the curvature.* Let H_n^* be the connected and simply connected Lie group associated to the Lie algebra $(\mathcal{H}_n^*, [,]^*)$ and let g be the left invariant Riemannian metric on H_n^* whose value at the identity is $\langle,\rangle_e^*$. The crucial fact is that the vanishing of the curvature of g is equivalent to the vanishing of the curvature of the Levi-Civita contravariant connection $\mathcal{D}$ associated to $(\pi, \langle,\rangle)$. Thus, if $\mathcal{D}$ is flat then, by Theorem 2.1, $\mathcal{H}_n^* = S_{\langle,\rangle^*} \oplus I$

where I is the derived ideal and I and $S_{\langle,\rangle^*}$ are abelian. Moreover, there exists $\lambda_1,\dots,\lambda_r \in S_{\langle,\rangle^*} \setminus \{0\}$ and $(\alpha_1,\beta_1,\dots,\alpha_r,\beta_r)$ an orthonormal basis of I such that, for any $s \in S_{\langle,\rangle^*}$ and for $i = 1,\dots,r$,

$$[s,\alpha_i]^* = \langle s,\lambda_i\rangle^* \beta_i \quad \text{and} \quad [s,\beta_i]^* = -\langle s,\lambda_i\rangle^* \alpha_i. \tag{8}$$

Put $v_i = \#(\lambda_i)$, $e_j = \#(\alpha_j)$ and $f_j = \#(\beta_j)$, where $\# : \mathcal{H}_n^* \to \mathcal{H}_n$ is the isomorphism induced by the metric. Now, we can deduce that $\#(S_{\langle,\rangle^*}) = \ker\xi$ and, for $i = 1,\dots,r$, $\xi(e_i) = -v_i \wedge f_i$ and $\xi(f_i) = v_i \wedge e_i$. To get (i) of Theorem 1.2, we will show that $\xi(z) = 0$ and $v_1,\dots,v_r$ are central elements of $\mathcal{H}_n$.

Suppose that $z \notin \ker\xi$. Then $\ker\xi$ is an abelian subalgebra of $\mathcal{H}_n$ and there exists $u_0 \in \ker\xi$ and $v_0 \in \ker\xi^\perp$ such that $[u_0,v_0] = z$. We have $\xi(z) = \xi([u_0,v_0]) = ad_{u_0}\xi(v_0) - ad_{v_0}\xi(u_0) = X \wedge z$, where

$$X = \sum_{i=1}^{r} \big(-\langle v_0,e_i\rangle_e \omega(u_0,f_i) + \langle v_0,f_i\rangle_e \omega(u_0,e_i)\big) v_i \in \ker\xi \setminus \{0\}.$$

Now, for any $u \in \ker\xi$ and for $i = 1,\dots,r$, we have

$$\omega(u,e_i) X \wedge z = \xi\big([u,e_i]\big) = ad_u\xi(e_i) = -\omega(u,f_i) v_i \wedge z,$$
$$\omega(u,f_i) X \wedge z = \xi\big([u,f_i]\big) = ad_u\xi(f_i) = \omega(u,e_i) v_i \wedge z,$$

and hence, since $z \notin \ker\xi$, $\omega(u,e_i)X + \omega(u,f_i)v_i = 0$ and $\omega(u,f_i)X - \omega(u,e_i)v_i = 0$. These relations imply that $\omega(u,e_i)^2 + \omega(u,f_i)^2 = 0$, thus $\omega(u,e_i) = \omega(u,f_i) = 0$ and then u is a central element which is in contradiction with $z \notin \ker\xi$. Thus $z \in \ker\xi$.

Now, by applying the 1-cocycle condition to (e_i,f_j) and (u_i,e_i), we get

$$0 = ad_{e_i}\xi(f_j) - ad_{f_j}\xi(e_i) = \omega(e_i,v_j) z \wedge e_j + \omega(e_i,e_j) v_j \wedge z + \omega(f_j,v_i) z \wedge f_i + \omega(f_j,f_i) v_i \wedge z,$$
$$0 = ad_u\xi(e_i) = -\omega(u,v_i) z \wedge f_i - \omega(u,f_i) v_i \wedge z.$$

We deduce from these relations that $\omega(e_i,v_j) = \omega(f_j,v_i) = \omega(u,v_i) = 0$ and hence $v_1,\dots,v_r$ are central elements of $\mathcal{H}_n$ and this achieves the proof of (i).

(ii) *The vanishing of the metacurvature.* In what follows, we identify $\mathcal{H}_n^*$ with the space of left invariant 1-forms where the bracket is the Koszul bracket, and we suppose that the curvature vanishes. We have $\mathcal{H}_n^* = S_{\langle,\rangle^*} \oplus I$ where I is the derived ideal and I and $S_{\langle,\rangle^*}$ are abelian. We will show now that metacurvature is given by the following formulas

$$\mathcal{M}(\alpha,\beta,\gamma) = \begin{cases} 0 & \text{if } \alpha, \quad \text{or } \beta, \quad \text{or } \gamma \in I, \\ -\big[\alpha,[\beta,d\gamma]_\pi\big]_\pi & \text{if } \alpha,\beta,\gamma \in S_{\langle,\rangle^*}. \end{cases} \tag{9}$$

For any $\alpha \in S_{\langle,\rangle^*}$, $\beta \in I$ and $\gamma \in \mathcal{H}_n^*$, one can deduce from (1) that

$$\mathcal{D}\alpha = 0, \quad \mathcal{D}_\alpha\gamma = [\alpha,\gamma]_\pi \quad \text{and} \quad \mathcal{D}_\beta\gamma = 0. \tag{10}$$

Note that, for any $\alpha \in I$, $d\alpha = 0$. Now, let $\alpha \in I$ and $\beta,\gamma \in \mathcal{H}_n^*$. Since α is closed, there exists f such that $\alpha = df$. Hence, from (3), $\mathcal{M}(\alpha,\beta,\gamma) = \{f,\{\beta,\gamma\}\} - \{\{f,\beta\},\gamma\} - \{\{f,\gamma\},\beta\}$. By applying (10), we get $\{f,\{\beta,\gamma\}\} = \mathcal{D}_{df}\{\beta,\gamma\} = 0$ ($\{\beta,\gamma\}$ is left invariant by (2)) and in a same way $\{f,\beta\} = \{f,\gamma\} = 0$. Thus $\mathcal{M}(\alpha,\beta,\gamma) = 0$.

Suppose now that $\alpha,\beta,\gamma \in S_{\langle,\rangle^*}$. From (10), α,β,γ are parallel and, from (4), $\mathcal{M}(\alpha,\beta,\gamma) = -\mathcal{D}_\beta\mathcal{D}_\gamma d\alpha \overset{(10)}{=} -[\beta,[\gamma,d\alpha]_\pi]_\pi$. Note that $\mathcal{M}$ is symmetric and (9) follows.

On the other hand, there exists $J : \mathcal{H}_n \to \mathcal{H}_n$ skew-symmetric with respect to $\langle,\rangle_e$ such that $J(z) = 0$ and, for any $u \in \mathcal{H}_n$, $\xi(u) = z \wedge Ju$. Since J is skew-symmetric, there exists $(z_1,\dots,z_l,e_1,f_1,\dots,e_r,f_r)$ an orthonormal family of $(\mathbb{R}z)^\perp$ and $(a_1,\dots,a_r) \in \mathbb{R}^l \setminus \{0\}$ such that $J(z_j) = 0$, $J(e_i) = a_i f_i$ and $J(f_i) = -a_i e_i$. Let $(\lambda,\lambda_1,\dots,\lambda_l,\alpha_1,\beta_1,\dots,\alpha_r,\beta_r)$ the dual basis of $(z,z_1,\dots,z_l,e_1,f_1,\dots,e_r,f_r)$. One can check easily that $S_{\langle,\rangle^*} = \mathrm{Vect}\{\lambda,\lambda_1,\dots,\lambda_l\}$, $\lambda_1,\dots,\lambda_l$ are central in $\mathcal{H}_n^*$, $d\lambda_1 = \dots = d\lambda_l = 0$ and, from (9), $i_{\lambda_i}\mathcal{M} = 0$, for $i = 1,\dots,l$. Thus $\mathcal{M}$ vanishes if and only if $[\lambda,[\lambda,d\lambda]_\pi]_\pi = 0$. To conclude, note that $d\lambda = -\omega$ and, for any left invariant 2-form ρ, $[\lambda,\rho]_\pi(u,v) = \rho(Ju,v) + \rho(u,Jv)$.

(iii) *The vanishing of the modular vector field.* The Riemannian volume form is given by

$$\mu = \lambda \wedge \lambda_1 \wedge \dots \wedge \lambda_l \wedge \alpha_1 \wedge \beta_1 \wedge \dots \wedge \alpha_r \wedge \beta_r$$

and, since $d\lambda_1 = \cdots = d\lambda_l = d\alpha_1 = d\beta_1 = \cdots = d\alpha_r = d\beta_r = 0$, one can see easily that for any $u \in \mathcal{H}_n$, $d(i_{\xi(u)}\mu) = 0$. Moreover, $\mathcal{H}_n$ and $\mathcal{H}_n^*$ are unimodular and, by Theorem 1.1, we deduce the vanishing of the modular vector field. This achieves the proof of Theorem 1.2. □

Let us give some examples.

Example 1. (i) The condition (ii) of Theorem 1.2 is always valid on H_1.

(ii) Let $\mathbb{B} = (z, e_1, f_1, \ldots, e_n, f_n)$ a Darboux basis of $\mathcal{H}_n$, i.e., z is a central element, $\omega(e_i, e_j) = \omega(f_i, f_j) = 0$ and $\omega(e_i, f_j) = \delta_{ij}$. Let $\lambda_1 < \lambda_2 < \cdots < \lambda_n$ a family of real numbers and consider any scalar product $\langle,\rangle_e$ for which $\mathbb{B}$ is an orthogonal basis. Define $J : \mathcal{H}_n \to \mathcal{H}_n$ by $J(z) = 0$, $J(e_i) = \lambda_i f_i$ and $J(f_i) = -\lambda_i e_i$. The endomorphism J is skew-symmetric with respect to $\langle,\rangle$ and one can check easily that J satisfies the condition (ii) of Theorem 1.2.

References

[1] M. Boucetta, Compatibilité des structures pseudo-riemanniennes et des structures de Poisson, C. R. Acad. Sci. Paris, Ser. I 333 (2001) 763–768.

[2] J.P. Dufour, N.T. Zung, Poisson Structures and Their Normal Forms, Progress in Mathematics, vol. 242, Birkhäuser Verlag, Basel, Boston, New York, 2005.

[3] R.L. Fernandes, Connections in Poisson geometry. I. Holonomy and invariants, J. Differential Geom. 54 (2) (2000) 303–365.

[4] E. Hawkins, Noncommutative rigidity, Commun. Math. Phys. 246 (2004) 211–235.

[5] E. Hawkins, The structure of noncommutative deformations, J. Differential Geom. 77 (2007) 385–424.

[6] J.L. Koszul, Crochet de Schouten–Nijenhuis et Cohomologie, Astérisque, Numéro Hors Série (1985) 257–271.

[7] J. Milnor, Curvature of Left invariant metrics on Lie groups, Adv. Math. 21 (1976) 293–329.

[8] I. Szymczak, S. Zakrzemski, Quantum deformations of the Heisenberg group obtained by geometric quantization, J. Geom. Phys. 7 (1990) 553–569.