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COMPTES RENDUS MATHEMATIQUE

Partial Differential Equations

# Nodal line structure of least energy nodal solutions for Lane–Emden problems

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**Abstract**

In this Note, we consider the Lane–Emden problem $-\Delta u = \lambda_2 |u|^{p-2} u$ with Dirichlet boundary conditions, where the domain $\Omega$ is an open bounded subset of $\mathbb{R}^2$, $\lambda_2$ is the second eigenvalue of $-\Delta$, and $p > 2$. We prove that, if $\Omega$ is $\mathcal{C}^2$ and convex, the nodal line intersects $\partial\Omega$ when $p$ is close to 2. In contrast, we also exhibit a connected — but not simply connected — domain $\Omega$ such that, for $p$ close to 2, the nodal line does not intersect $\partial\Omega$. ***To cite this article: C. Grumiau, C. Troestler, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***


**Résumé**

**Structure de la ligne nodale des solutions nodales d'énergie minimale pour le problème de Lane–Emden.** Soit l'équation $-\Delta u = \lambda_2 |u|^{p-2} u$ avec conditions au bord de Dirichlet, où $\Omega \subseteq \mathbb{R}^2$ est ouvert borné, $\lambda_2$ la deuxième valeur propre de $-\Delta$ et $p > 2$. Nous prouvons que, sur un convexe de classe $\mathcal{C}^2$, la ligne nodale de toute solution nodale d'énergie minimale intersecte $\partial\Omega$ pour $p$ proche de 2. Par ailleurs, nous montrons également l'existence d'un ensemble connexe mais non simplement connexe, tel que, pour $p$ proche de 2, la ligne nodale de toute solution nodale d'énergie minimale n'intersecte pas $\partial\Omega$. ***Pour citer cet article : C. Grumiau, C. Troestler, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***


## Version française abrégée

Dans [5] et [9], les auteurs ont obtenu que, pour $p$ proche de 2, les solutions nodales d'énergie minimale de l'équation $-\Delta u = \lambda_2 |u|^{p-2} u$, avec conditions au bord de Dirichlet sur un domaine ouvert et borné $\Omega$, ont les mêmes symétries que leur projection sur le deuxième espace propre de $-\Delta$. Ceci implique en particulier que leur ligne nodale est un diamètre pour les disques et les anneaux, une médiane pour les rectangles, … Nous étendons ici le fait que la ligne nodale des solutions nodales d'énergie minimale touche le bord pour des domaines convexes sans besoin de symétrie. De plus nous montrons que l'hypothèse de convexité ne peut être enlevée.

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Les ingrédients sont les suivants. Premièrement, dans le cas linéaire $-\Delta u = \lambda_2 u$, il est connu [2,12] que la ligne nodale d'une seconde fonction propre de $-\Delta$ intersecte $\partial\Omega$ en deux points si $\Omega$ est convexe. On sait également [10] qu'il existe un domaine connexe (mais non simplement connexe) $\Omega$ tel que $\lambda_2$ est simple et la ligne nodale de la deuxième fonction propre n'intersecte pas $\partial\Omega$. Ensuite, nous prouvons que, quand $\Omega$ est de classe $\mathscr{C}^2$ et $p \to 2$, les points d'accumulation faibles (pour la norme $H_0^1$) $u_2$ d'une famille $(u_p)_{p>2}$ de solutions nodales d'énergie minimale sont aussi des points d'accumulation forts pour la norme $\mathscr{C}^1$. Nous en déduisons que si la ligne nodale de $u_2$ intersecte (resp. n'intersecte pas) $\partial\Omega$, la ligne nodale de $u_p$ en fait de même pour $p$ suffisamment proche de 2. Remarquons que la ligne nodale étant de mesure nulle, une convergence $H_0^1$ (et donc presque partout) n'est pas suffisante pour obtenir ces résultats.

## 1. Introduction

We consider the super-linear elliptic boundary value problem

$$\begin{cases} -\Delta u = \lambda_2 |u|^{p-2} u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{$\mathscr{P}_p$}$$

where $\Omega \subseteq \mathbb{R}^2$ is an open bounded domain and $p > 2$. We write $\lambda_i = \lambda_i(\Omega)$ for the $i$th distinct eigenvalue of $-\Delta$, and $E_i$ for the $i$th eigenspace. The usual norm in $H_0^1(\Omega)$ is denoted $\|\cdot\|$ and the norm in $L^p(\Omega)$ is $\|\cdot\|_{L^p}$. Let $B(v, r)$ be the open ball $\{u \in H_0^1(\Omega) \mid \|u - v\| < r\}$ and $B[v, r]$ the closed ball. The symbol $\Subset$ denotes a compact embedding.

It is well known that, for $p > 2$, problem ($\mathscr{P}_p$) has a positive ground state solution [3]. B. Gidas, W.N. Ni and L. Nirenberg [8] showed, using the elegant and now celebrated moving planes technique, that, on a convex domain, the ground state inherits the symmetries of the domain.

A. Castro, J. Cossio and J.M. Neuberger [7] proved the existence of a nodal solution with least energy among nodal solutions, which is therefore referred to as a *least energy nodal solution* of problem ($\mathscr{P}_p$). This solution has two nodal domains, just as the second eigenfunctions of $-\Delta$. Whereas ground state solutions inherit the symmetries of the domain, A. Aftalion and F. Pacella [1] proved in 2004 that, on a ball, a least energy nodal solution cannot be radial and the nodal line always intersects the boundary of the domain. On the other hand, in 2005, T. Bartsch, T. Weth and M. Willem [4] obtained partial symmetry results: they showed that, on a radial domain, a least energy nodal solutions $u$ have the so-called Schwarz foliated symmetry, i.e. $u$ can be written as $u(x) = \tilde{u}(|x|, \xi \cdot x)$, where $\xi \in \mathbb{R}^2$ and $\tilde{u}(r, \cdot)$ is nondecreasing for every $r > 0$. It does not directly imply that the nodal line is a diameter let alone intersects the boundary. In 2007, F. Pacella and T. Weth [14] proved that, on a radial domain, solutions with Morse index less than 2 respect the Schwarz foliated symmetry and their nodal line intersects $\partial\Omega$.

In this Note, we establish that for a convex domain $\Omega \subseteq \mathbb{R}^2$ not necessarily possessing any symmetry, the zero set of least energy nodal solutions intersects $\partial\Omega$. Our proofs are inspired from a recent work in collaboration with D. Bonheure, V. Bouchez and J. Van Schaftingen [5,9], where we proved that, for $p$ close to 2, least energy nodal solutions possess the symmetries of their projection in the second eigenspace $E_2$. In the same vein, we establish that a family $(u_p)_{p>2}$ of least energy nodal solutions of ($\mathscr{P}_p$) converges, in a suitable sense, to some $u_2 \in E_2$ and so that the zero set of $u_p$ is close to the zero set of $u_2$. We then conclude using a result by G. Alessandrini [2] (see also [12]) saying that the nodal line of the non-zero second eigenfunctions of $-\Delta$ intersects $\partial\Omega$ at exactly two points when $\Omega \subseteq \mathbb{R}^2$ is convex. Using a theorem by M. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof and N. Nadirashvili [10], we also show that, if the convexity assumption is removed, the zero set may well not touch $\partial\Omega$ anymore. It is a conjecture that, on a simply connected domain, the nodal line always intersects $\partial\Omega$. However, at this time, this is not even proven to be true for the linear case.

The Note is organised as follows: As the family of least energy nodal solutions $(u_p)_{p>2}$ is bounded (see [5]), without loss of generality, we can assume that $u_p \rightharpoonup u_2 \in E_2$, up to a subsequence. In Section 2, we prove that, when $\Omega$ is of class $\mathscr{C}^2$ and $p \to 2$, weak accumulation points of the family $(u_p)_{p>2}$ in the $H_0^1$-norm are strong accumulation points in $E_2$ for the $\mathscr{C}^1$-norm. Section 3 then uses this result to establish that the nodal line of $u_p$ intersects (resp. does not intersect) the boundary of $\Omega$ when the nodal line of $u_2$ does (resp. does not).

## 2. Convergence in $\mathscr{C}^1$

Let $(u_p)_{p>2}$ be a family of least energy nodal solutions of $(\mathscr{P}_p)$. D. Bonheure et al. [5] proved that $(u_p)_{p>2}$ is bounded in $H_0^1(\Omega)$, stays away from zero, and that weak accumulation points for the $H_0^1$-norm are in fact strong accumulation points and lie in $E_2$. Here we show that they also are accumulation points for $\mathscr{C}^1$-norm.

**Lemma 1.** *For any sequence $(p_n) \subseteq ]2, +\infty[$, if $p_n \to 2$ and $u_{p_n} \rightharpoonup u_2$ in $H_0^1$, then $|u_{p_n}|^{p_n-2}u_{p_n} \to u_2$ in $L^q$, for all $q > 1$.*

**Proof.** Let $q > 1$, $\bar{p} := \sup_{n\in\mathbb{N}} p_n$ and $r := q(\alpha - 1) > 1$. Given the result recalled before [5] and the Sobolev embedding theorem, one can assume $u_{p_n} \to u_2$ in $L^r$, for all $r > 1$. Thus, taking if necessary a subsequence, there exists $g \in L^r$ such that, almost everywhere, $|u_{p_n} - u_2| \leqslant g$ (see e.g. [15, Proposition 14.9]). As $u_2 \in \mathscr{C}(\bar{\Omega})$, there exists a constant $K$ such that, a.e., $|u_{p_n}| \leqslant g + K$. Therefore, for all $n \in \mathbb{N}$,

$$\big||u_{p_n}|^{p_n-2}u_{p_n} - u_2\big| \leqslant \max(1, g + K)^{\bar{p}-1} + K \in L^{r/(\bar{p}-1)}$$

and so $|u_{p_n}|^{p_n-1}u_{p_n} - u_2 \in L^q$. Using Lebesgue's dominated convergence theorem and the fact that the limit does not depend on the subsequence, we can conclude. □

**Proposition 2.** *If $\Omega$ is of class $\mathscr{C}^2$ and $u_2 \in E_2$ is a weak accumulation point of $(u_p)$ in $H_0^1$, $u_2$ is a accumulation point in $\mathscr{C}^1(\bar{\Omega})$.*

**Proof.** By hypothesis, there exists a sequence $2 < p_n \to 2$ such that $u_{p_n} \rightharpoonup u_2$ in $H_0^1$. We have, for all $n \in \mathbb{N}$,

$$\begin{cases} -\Delta(u_{p_n} - u_2) = \lambda_2\big(|u_{p_n}|^{p_n-2}u_{p_n} - u_2\big), & \text{in } \Omega, \\ u_{p_n} - u_2 = 0, & \text{on } \partial\Omega. \end{cases}$$

By Lemma 1, $|u_{p_n}|^{p_n-2}u_{p_n} \to u_2$ in $L^q$, for all $q > 1$. Elliptic regularity estimates [6] imply that, for $p_n$ sufficiently close to 2, $u_{p_n} \in W^{2,q}$ and $u_{p_n} \to u_2$ in $W^{2,q}$. Recall that $W^{k,q}(\Omega) \Subset \mathscr{C}^{m,\alpha}(\bar{\Omega})$ when $m < k - 2/q$ and $0 \leqslant \alpha < \lfloor k - \frac{2}{q} - m \rfloor$. Thus, taking $q$ sufficiently large so that $W^{2,q} \Subset \mathscr{C}^1$, we conclude that $u_{p_n} \to u_2$ in $\mathscr{C}^1$. □

**Remark 3.** The above two results hold in dimensions $N < 2^*$ (i.e., $N = 2$ or 3).

## 3. Nodal line structure

### *3.1. General asymptotic results*

In this section, we call the $\varepsilon$-neighbourhood of a set $A$ the set of points $x \in \Omega$ such that the distance between $x$ and $A$ is less than $\varepsilon$. For any $u \in \mathscr{C}^0(\Omega)$ with two nodal domains, let $\mathcal{N}(u) := \overline{\{x \in \Omega \mid u = 0\}}$ denote its nodal set, $\mathrm{ND}_u^+$ its positive nodal domain, and $\mathrm{ND}_u^-$ its negative nodal domain.

**Proposition 4.** *Let $\Omega$ be a domain of class $\mathscr{C}^2$ and $u_2 \in E_2$ be such that $\overline{\mathrm{ND}_{u_2}^+} \setminus \mathcal{N}(u_2)$ and $\overline{\mathrm{ND}_{u_2}^-} \setminus \mathcal{N}(u_2)$ intersect the same connected component of $\partial\Omega$. If $u_{p_n} \rightharpoonup u_2$ in $H_0^1$, then, for $p_n$ close to 2, $\mathcal{N}(u_{p_n})$ intersects $\partial\Omega$.*

**Proof.** By contradiction, let us assume that there exists a subsequence, still denoted $p_n$, such that $p_n \to 2$ and the nodal sets of $u_{p_n}$ do not intersect $\partial\Omega$. Let $\Gamma$ be a connected component of $\partial\Omega$ that both $\overline{\mathrm{ND}_{u_2}^+} \setminus \mathcal{N}(u_2)$ and $\overline{\mathrm{ND}_{u_2}^-} \setminus \mathcal{N}(u_2)$ intersect. Since $\mathcal{N}(u_{p_n})$ stays away from $\Gamma$, $u_{p_n}$ has always the same sign in a neighbourhood of $\Gamma$. Going if necessary to a subsequence, we can assume that $u_{p_n} > 0$ in a neighbourhood of $\Gamma$ for all $n$ (the case $u_{p_n} < 0$ can be treated similarly). Hopf's lemma implies $\partial u_{p_n}/\partial\nu < 0$ for all $x \in \Gamma$, where $\partial/\partial\nu$ is the derivative in the outer normal direction. Pick $x \in \Gamma \cap (\overline{\mathrm{ND}_{u_2}^-} \setminus \mathcal{N}(u_2))$. Since $\mathcal{N}(u_2)$ is compact, there exist a connected neighbourhood $U$ of $x$ such that $u_2 < 0$ in $U \cap \Omega$. Thus, by Hopf's lemma, $\partial u_2/\partial\nu > 0$. As $u_{p_n} \to u_2$ in $\mathscr{C}^1(\bar{\Omega})$ by Proposition 2, this is a contradiction. □

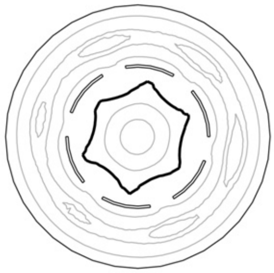

Fig. 1. Level curves of $u_2$. The nodal line is in bold.

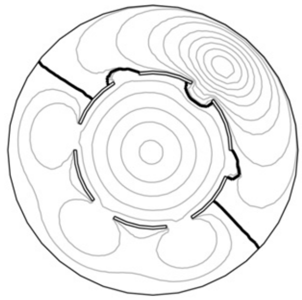

Fig. 2. Level curves of $u_3$. The nodal line is in bold.

**Proposition 5.** *Let $\Omega$ be an open bounded domain of class $\mathscr{C}^2$. If $u_p \to u_2$ and the nodal line of $u_2$ does not intersect $\partial\Omega$, then, for $p$ close to 2, the nodal line of $u_p$ does neither intersect $\partial\Omega$.*

**Proof.** As $\mathcal{N}(u_2)$ does not intersect $\partial\Omega$ which is compact, there exists $\varepsilon > 0$ such that $u_2$ does not vanish in an $\varepsilon$-neighbourhood of $\partial\Omega$. Thus Hopf's lemma and the compactness of $\partial\Omega$ imply the existence of a constant $c > 0$ such that $|\partial u_2(x)/\partial\nu| \geqslant c$ for all $x \in \partial\Omega$. Using the fact that $u_p \to u_2$ in $\mathscr{C}^1$ (Proposition 2), we conclude that, for $p$ sufficiently close to 2, $|\partial u_p/\partial\nu| \geqslant c/2$ on $\partial\Omega$ and therefore $u_p$ does not vanish in a neighbourhood of $\partial\Omega$. □

**Remark 6.** The boundary of a simply connected domain of $\mathbb{R}^2$ has a unique connected component [11]. Note also that, thanks to Remark 3, Propositions 4 and 5 are also valid in dimension $N = 3$.

### *3.2. The convex case*

In 1994, G. Alessandrini [2] showed that, when $\Omega$ is convex, the nodal line of the second eigenfunctions of $-\Delta$ always intersects $\partial\Omega$ at exactly 2 points. Using this and Proposition 4, we immediately obtain:

**Theorem 7.** *On a convex domain of class $\mathscr{C}^2$, for $p$ sufficiently close to 2, the nodal line of least energy nodal solutions $u_p$ intersects the boundary of $\Omega$.*

### *3.3. The non-simply connected case*

In 1995, M. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof and N. Nadirashvili [10] exhibited a connected (but not simply connected) domain $\Omega$ such that the second eigenvalue of $\Delta$ is simple and the nodal line of a second eigenfunction does not intersect $\partial\Omega$. To describe it, let us work in polar coordinates $x = r(\cos\omega, \sin\omega)$ with $0 \leqslant \omega < 2\pi$ and select $0 < R_1 < R_2 < R_3$ such that $\lambda_1(B(0, R_1)) < \lambda_1(A(R_2, R_3)) < \lambda_2(B(0, R_1))$, where $A(R_2, R_3)$ denotes the annulus $B(0, R_3) \setminus B[0, R_2]$. Then consider

$$D_{b,\varepsilon} := B(0, R_1) \cup A(R_2, R_3) \cup \bigcup_{j=0}^{b-1} \left\{ x \in \mathbb{R}^2\colon\ R_1 \leqslant r \leqslant R_2,\ \left| \omega - \frac{2\pi j}{b} \right| \bmod 2\pi < \varepsilon \right\},$$

which is a disc surrounded by an annulus joined by $b$ small bridges. For $b$ sufficiently large and $\varepsilon$ sufficiently small, M. Hoffmann-Ostenhof et al. show that the nodal line of $u_2$ does not intersect $\partial D_{b,\varepsilon}$. Since their proof does not use the structure of the bridges between $B(0, R_1)$ and $A(R_2, R_3)$ but only the group of reflections that leaves $D_{b,\varepsilon}$ invariant and the fact that the bridges are small, we can smooth the domain so that it is of class $\mathscr{C}^2$. By Proposition 5, we then conclude:

**Theorem 8.** *There exists connected domain such that, for $p$ close to 2, the nodal sets of the least energy nodal solutions do not intersect the boundary of $\Omega$.*

Fig. 1 shows the level curves of a second eigenfunction of $-\Delta$ for $b = 6$ bridges. You can see that the nodal line is included into the ball $B(0, R_1)$ and that the second eigenfunction $u_2$ is even with respect to any reflection leaving $\Omega$

invariant [10, Lemma 1]. Therefore, by [5], $u_p$ is even with respect to any reflection that leaves $\Omega$ invariant, for $p$ close to 2.

Numerical experiments seem to indicate that Theorem 8 and the above symmetry properties of $u_p$ do not remain valid for values of $p$ farther from 2. For example, Fig. 2 represents levels curves of $u_p$ for $p = 3$ computed using the so called "modified mountain pass algorithm" proposed by J.M. Neuberger [13]. It clearly shows that $\mathcal{N}(u_3)$ touches $\partial\Omega$ and that $u_3$ is only even with respect to a single reflection.

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