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COMPTES RENDUS MATHEMATIQUE

Numerical Analysis

Ready-to-blossom bases and the existence of geometrically continuous piecewise Chebyshevian B-splines

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Abstract

Existence of blossoms is crucial for design. In a single space, we recently characterised it in terms of *ready-to-blossom bases*. Such bases are magic, for their use makes existence of blossoms visible at first sight. A similar characterisation is given here for geometrically continuous piecewise Chebyshevian splines (sections in different Extended Chebyshev spaces, connection matrices at the knots). This enables us to re-prove the equivalence between existence of blossoms and existence of B-spline bases under the least possible differentiability assumptions. The existing proof of the latter result was totally different and it strongly relied on the fact that all spline sections were supposed to be C^∞. ***To cite this article: M.-L. Mazure, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

B-splines de Chebyshev et bases sur mesure pour les floraisons. Dans tout espace de splines à sections dans différents espaces de Chebyshev généralisés et matrices de connexion, nous caractérisons l'existence de floraisons (cruciale pour le design) par celle de bases *sur mesure* définies en termes de zéros. Ce résultat nous permet d'obtenir l'équivalence entre l'existence de floraisons et celle de bases de B-splines sous des hypothèses de différentiabilité aussi faibles que possible. ***Pour citer cet article : M.-L. Mazure, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Version française abrégée

Le théorème suivant caractérise les espaces de splines de Chebyshev pertinents pour le design :

Théorème 0.1. *Dans tout espace $\mathbb{S}$ de splines géométriquement continues à sections dans différents espaces de Chebyshev généralisés, les propriétés suivantes sont équivalentes* :

(1) *les floraisons existent dans* $\mathbb{S}$;

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(2) *l'espace $\mathbb{S}$ possède une base de B-splines, et il en est de même de tout espace de splines obtenu à partir de $\mathbb{S}$ par insertions de nœuds.*

Cette équivalence fut établie dans [1] pour des espaces à sections C^∞, hypothèse qui intervenait très fortement dans la démonstration. Grâce à la notion de *bases sur mesure pour les floraisons*, nous en donnons ici une toute autre preuve qui permet de travailler sous une hypothèse de différentiabilté minimum (voir Section 1). La difficulté principale consiste dans tous les cas à montrer que, dès que les floraisons existent, elles sont *pseudoaffines* en chaque variable sur leur domaine de définition. Cette propriété essentielle est à la base de tous les algorithmes de *design* tels *l'algorithme de Boor* qui génère automatiquement une base de B-splines. Elle sera ici obtenue à partir d'une version spline du résultat suivant (voir Th. 2.1) :

Théorème 0.2. *(Voir [2].) Soit $\mathbb{E} \subset C^n(I)$ un W-espace de dimension $(n+1)$ (i.e., le Wronskien d'une base de $\mathbb{E}$ ne s'annule pas sur I) contenant les constantes. Pour tout $x \in I$ et tout $k \leqslant n$, on notera Ψ_k^x un élément de $\mathbb{E}$ qui s'annule exactement k fois en x. Supposons que les floraisons existent dans $\mathbb{E}$. Alors, pour tous entiers positifs $\mu_1, \ldots, \mu_r$ de somme n, et pour tous $a_1, \ldots, a_r \in I$ deux à deux distincts, les fonctions $\mathbb{1}, \Psi_n^{a_1}, \ldots, \Psi_{n-\mu_1+1}^{a_1}, \ldots, \Psi_n^{a_r}, \ldots, \Psi_{n-\mu_r+1}^{a_r}$ forment une base de $\mathbb{E}$.*

Rappelons que, dans le cas ci-dessus, les floraisons sont des fonctions de n variables définies géométriquement sur I^n à partir d'intersections de variétés osculatrices. Considérons une fonction $\Phi := (\Psi_n^{a_1}, \ldots, \Psi_{n-\mu_1+1}^{a_1}, \ldots, \Psi_n^{a_r}, \ldots, \Psi_{n-\mu_r+1}^{a_r}) : I \to \mathbb{R}^n$. Si ses composantes forment avec la fonction constante $\mathbb{1}$ une base de $\mathbb{E}$, il est facile de vérifier que l'intersection des variétés osculatrices de Φ, d'ordre $(n-\mu_i)$ au point a_i, $1 \leqslant i \leqslant r$, se réduit alors à l'origine de $\mathbb{R}^n$. Cette remarque montre que la réciproque du Théorème 0.2 est évidente, ce qui justifie le qualificatif "sur mesure pour les floraisons" que nous attribuons à de telles bases.

Dans le cas de splines dont les espaces-sections sont de dimension $(n+1)$, le domaine de définition naturel des floraisons est un sous-ensemble restreint de n-uplets, dits admissibles. Seuls ces n-uplets donneront alors naissance à de telles bases (Théorème 2.1).

1. The context – The result

1.1. EC-spaces and ready-to-blossom bases

Let I be a non-trivial interval and let $\mathbb{F} \subset C^n(I)$ be an $(n+1)$-dimensional space. We say that $\mathbb{F}$ is *a W-space on I* if the Wronskian of a basis of $\mathbb{F}$ does not vanish on I. We say that $\mathbb{F}$ is *an Extended Chebyshev space* (*EC-space*) *on I* if any non-zero element of $\mathbb{F}$ vanishes at most n times in I, counting multiplicities up to $(n+1)$. An EC-space on I is a W-space on I, the converse being false in general.

Throughout the present section we assume that $\mathbb{F}$ is a given $(n+1)$-dimensional W-space on I which contains constants. Choose any non-degenerate function $\Phi = (\Phi_1, \ldots, \Phi_n) \in \mathbb{F}^n$, that is, choose any n functions $\Phi_1, \ldots, \Phi_n \in \mathbb{F}$ so that $(\mathbb{1}, \Phi_1, \ldots, \Phi_n)$ forms a basis of $\mathbb{F}$. Then Φ is a *mother-function* of the space $\mathbb{F}$, from which any function $F \in \mathbb{F}^d$ (for any $d \geqslant 1$) can be obtained via an affine map. The *osculating flat* of order i $(0 \leqslant i \leqslant n)$ at a point $x \in I$ is the affine flat defined by the point $\Phi(x)$ and the span of the i first derivatives of Φ at x. We denote it by $\mathrm{Osc}_i\,\Phi(x)$. It is i-dimensional. We say that *blossoms exist in the space* $\mathbb{F}$ if, for any $r \geqslant 1$, any pairwise distinct $a_1, \ldots, a_r$ in I and any positive integers $\mu_1, \ldots, \mu_r$ summing to n, the osculating flats $(\mathrm{Osc}_{n-\mu_i}\,\Phi(a_i))_{i=1}^r$ intersect at a single point. The *blossom* φ of Φ is then the symmetric function

$$\varphi = (\varphi_1, \ldots, \varphi_n) : I^n \longrightarrow \mathbb{R}^n, \quad \bigl\{\varphi\bigl(a_1^{[\mu_1]}, \ldots, a_r^{[\mu_r]}\bigr)\bigr\} = \bigcap_{i=1}^r \mathrm{Osc}_{n-\mu_i}\,\Phi(a_i), \tag{1}$$

where, for any $t \in \mathbb{R}$ and any non-negative integer k, $t^{[k]}$ stands for a k-fold repetition of the point t. The blossom of any $F \in \mathbb{F}^d$ is then obtained from φ via the affine map applying Φ on F.

For each $x \in I$, and each integer k, $0 \leqslant k \leqslant n$, one can choose a function $\Psi_k^x \in \mathbb{F}$ which vanishes exactly k times at x.

Definition 1.1. Assume that, up to permutation, a given n-tuple $(x_1,\dots,x_n)\in I^n$ is equal to $(a_1^{[\mu_1]},\dots,a_r^{[\mu_r]})$, with positive integers $\mu_1,\dots,\mu_r$ and pairwise distinct $a_1,\dots,a_r$. Then, if the sequence $(\mathbb{1},\Psi_n^{a_1},\dots,\Psi_{n-\mu_1+1}^{a_1},\Psi_n^{a_2},\dots,\Psi_n^{a_r},\dots,\Psi_{n-\mu_r+1}^{a_r})$ forms a basis of $\mathbb{F}$, we say that it is *a ready-to-blossom basis relative to* $(x_1,\dots,x_n)$.

The use of a mother-function derived from a ready-to-blossom basis relative to the n-tuple $(a_1^{[\mu_1]},\dots,a_r^{[\mu_r]})$ makes it obvious that the intersection (1) reduces to a single point. This justifies our terminology. In [2] we proved the following result:

Theorem 1.2. *Let $\mathbb{F}$ be an $(n+1)$-dimensional W-space on I, containing constants, with $n\geqslant 2$. The following properties are equivalent*

(1) *blossoms exist in $\mathbb{F}$;*
(2) *the space $D\mathbb{F}:=\{DF:=F^{(1)}:=F' \mid F\in\mathbb{F}\}$ is an (n-dimensional) EC-space on I;*
(3) *the space $\mathbb{F}$ possesses ready-to-blossom bases relative to any n-tuple $(x_1,\dots,x_n)\in I^n$.*

Assume that blossoms exist in $\mathbb{F}$. Then, in any ready-to-blossom basis relative to a given $(x_1,\dots,x_n)\in I^n$, the first coordinate of any $F\in\mathbb{F}$ is the value $f(x_1,\dots,x_n)$ of its blossom f at $(x_1,\dots,x_n)$. Furthermore, for any pairwise distinct $a_1,\dots,a_r\in I$ and any positive integers $\mu_1,\dots,\mu_r$ such that $\sum_{i=1}^r \mu_i<n$, we have

$$W\big(\Psi_n^{a_1\prime},\dots,\Psi_{n-\mu_1+1}^{a_1\prime},\Psi_n^{a_2\prime},\dots,\Psi_n^{a_r\prime},\dots,\Psi_{n-\mu_r+1}^{a_r\prime}\big)(x)\neq 0,\quad \textit{for all } x\in I\setminus\{a_1,\dots,a_r\}. \tag{2}$$

Note that when $n=1$, (1) is always satisfied, while we still have (2) $\Leftrightarrow$ (3).

1.2. Spline spaces, B-spline bases, and blossoms

Suppose that $n\geqslant 2$ is given. Let us start with

- a bi-infinite sequence of knots t_k, $k\in\mathbb{Z}$, satisfying $t_k<t_{k+1}$ for all k;
- a bi-infinite sequence of associated integers m_k, $k\in\mathbb{Z}$, satisfying $0\leqslant m_k\leqslant n$ for all k: m_k is the multiplicity of the knot t_k;
- a bi-infinite sequence of connection matrices M_k, $k\in\mathbb{Z}$: for each k, M_k is a square matrix of order $n-m_k$ which is lower-triangular and which has positive diagonal elements;
- a bi-infinite sequence of section spaces $\mathbb{E}_k$, $k\in\mathbb{Z}$: for each k, $\mathbb{E}_k\subset C^n([t_k,t_{k+1}])$ contains constants and the space $D\mathbb{E}_k:=\{DF:=F' \mid F\in\mathbb{E}_k\}$ is an n-dimensional EC-space on $[t_k,t_{k+1}]$.

The first two ingredients give birth to the knot vector $\mathbb{K}:=(t_k{}^{[m_k]})_{k\in\mathbb{Z}}$. Based on the latter data, we consider the set $\mathbb{S}$ of all *geometrically continuous piecewise Chebyshevian splines*, *i.e.*, all continuous functions S: $I:=\,]\operatorname{Inf}_{k\in\mathbb{Z}} t_k,\operatorname{Sup}_{k\in\mathbb{Z}} t_k[\,\to\mathbb{R}$ such that

(1) for each $k\in\mathbb{Z}$, the restriction of S to $[t_k,t_{k+1}]$ belongs to $\mathbb{E}_k$;
(2) for each $k\in\mathbb{Z}$, the following connection condition is fulfilled

$$\big(S^{(1)}(t_k^+),\dots,S^{(n-m_k)}(t_k^+)\big)^T=M_k\cdot\big(S^{(1)}(t_k^-),\dots,S^{(n-m_k)}(t_k^-)\big)^T.$$

The spline space $\mathbb{S}$ contains constants. It is essential to guarantee existence of B-splines in $\mathbb{S}$, defined as below. From now on, for the sake of simplicity, we assume the knot vector $\mathbb{K}$ to be bi-infinite, and we also denote it as $\mathbb{K}=(\xi_\ell)_{\ell\in\mathbb{Z}}$. Other cases would require slight changes in the definition below (see [1]):

Definition 1.3. We say that a sequence N_ℓ, $\ell\in\mathbb{Z}$, of elements in the spline space $\mathbb{S}$ is *the B-spline basis of* $\mathbb{S}$ when it meets the following requirements:

$(\mathrm{BSB})_1$ (*support property*) for each $\ell\in\mathbb{Z}$, $\operatorname{supp} N_\ell=[\xi_\ell,\xi_{\ell+n+1}]$;

$(\mathrm{BSB})_2$ (*end-point property*) for each $\ell \in \mathbb{Z}$, N_ℓ vanishes exactly $n - s + 1$ times at ξ_ℓ and exactly $n - s' + 1$ times at $\xi_{\ell+n+1}$, where $s := \sharp\{j \geqslant \ell \mid \xi_j = \xi_\ell\}$ and $s' := \sharp\{j \leqslant \ell + n + 1 \mid \xi_j = \xi_{\ell+n+1}\}$;
$(\mathrm{BSB})_3$ (*positivity property*) for each $\ell \in \mathbb{Z}$, N_ℓ is positive in the interior of its support;
$(\mathrm{BSB})_4$ (*normalisation property*) $\sum_{\ell\in\mathbb{Z}} N_\ell(x) = 1$ for all $x \in I$.

The possible existence of blossoms in the spline space $\mathbb{S}$ can be more easily perceived with the help of a relevant $(n + 1)$-dimensional linear space $\mathbb{E}$ contained in $\mathbb{S}$. As is classical, we complete each connection matrix M_k into a lower triangular matrix $\widehat{M}_k$ or order n, with positive diagonal elements. The space $\mathbb{E}$ is then composed of all splines $S \in \mathbb{S}$ which satisfy the stronger connection conditions:

$$\bigl(S^{(1)}(t_k^+), \ldots, S^{(n)}(t_k^+)\bigr)^T = \widehat{M}_k \cdot \bigl(S^{(1)}(t_k^-), \ldots, S^{(n)}(t_k^-)\bigr)^T, \quad k \in \mathbb{Z}.$$

Choose a mother-function $\Phi := (\Phi_1, \ldots, \Phi_n) \in \mathbb{E}^n$. Due the nature of the connection matrices, for any $x \in I$, and for $0 \leqslant i \leqslant n$, the osculating flat $\mathrm{Osc}_i\, \Phi(x)$ is always well defined, with the meaning of either $\mathrm{Osc}_i\, \Phi(x^-)$ or $\mathrm{Osc}_i\, \Phi(x^+)$ when x is a knot t_k (see [1]). The main difference between the non-spline case of a single space and the spline case is in the domain of definition of blossoms: while in the former case blossoms are defined on the whole of I^n, in the latter case their natural domain of definition is the set $\mathcal{A}_n(\mathbb{K})$ of all *admissible n-tuples* (relative to the knot vector $\mathbb{K}$),

$$\begin{aligned}\mathcal{A}_n(\mathbb{K}) = \bigl\{&(x_1, \ldots, x_n) \in I^n \mid \forall i \in \mathbb{Z}\colon \min(x_1, \ldots, x_n) < t_i < \max(x_1, \ldots, x_n)\\ &\Rightarrow t_i \text{ appears at least } m_i \text{ times among } (x_1, \ldots, x_n)\bigr\}.\end{aligned} \tag{3}$$

We say that *blossoms exist in the spline space* $\mathbb{S}$ if, for any pairwise distinct $a_1, \ldots, a_r$ in I and any positive integers $\mu_1, \ldots, \mu_r$ such that $\sum_{i=1}^r \mu_i = n$ and such that the n-tuple $(a_1^{[\mu_1]}, \ldots, a_r^{[\mu_r]})$ is admissible (*i.e.*, it belongs to $\mathcal{A}_n(\mathbb{K})$), the osculating flats $(\mathrm{Osc}_{n-\mu_i}\, \Phi(a_i))_{i=1}^r$ intersect at a single point. The *blossom* of the spline $\Phi \in \mathbb{S}^n$ is then the symmetric function φ defined as in (1), but now only on the symmetric set $\mathcal{A}_n(\mathbb{K})$. To see how to deduce the blossom of any spline $S \in \mathbb{S}^d$ from φ via affine maps, we refer to [1].

Consider another spline space $\bar{\mathbb{S}}$ defined on I, with $(n + 1)$-dimensional section spaces, and let $\bar{\mathbb{K}}$ denote its knot vector. Then, $\bar{\mathbb{S}}$ contains $\mathbb{S}$ if and only if any knot t_k appears in $\bar{\mathbb{K}}$ with a multiplicity greater than or equal to m_k, and the section spaces for $\bar{\mathbb{S}}$ are restrictions of those of $\mathbb{S}$. It is why $\bar{\mathbb{S}}$ is said to be obtained from $\mathbb{S}$ by knot insertion. We shall prove the following result, already proved in [1] under the assumption $\mathbb{E}_k \subset C^\infty([t_k, t_{k+1}])$ for all $k \in \mathbb{Z}$. This will be done via a spline version of Theorem 1.2 which is of interest in itself (see Section 2).

Theorem 1.4. *In the spline space $\mathbb{S}$ described above, the following two statements are equivalent*:

(1) *blossoms exist in* $\mathbb{S}$;
(2) *there exists a B-spline basis in* $\mathbb{S}$ *and in any spline space derived from* $\mathbb{S}$ *by knot insertion.*

2. Ready-to-blossom bases and existence of blossoms in $\mathbb{S}$

Throughout this section we are in the situation described in Section 1.2. For each $x \in I$, and each integer k, $0 \leqslant k \leqslant n$, one can choose a function $\Psi_k^x \in \mathbb{E}$ which vanishes exactly k times at x. This is due both to the fact that each space $\mathbb{E}_k$ is an EC-space on $[t_k, t_{k+1}]$ and to the structure of all matrices $\widehat{M}_k$. One can thus define possible ready-to-blossom bases in $\mathbb{E}$ similarly to Definition 1.1. For any integer $p \geqslant 1$, we define the set $\mathcal{A}_p(\mathbb{K})$ of all admissible p-tuples similarly to (3). For any p-tuple $(x_1, \ldots, x_p) \in \mathcal{A}_p(\mathbb{K})$ we denote by $\mathcal{J}(x_1, \ldots, x_p)$ the set of all $x \in I$ such that $(x_1, \ldots, x_p, x)$ is admissible (*i.e.*, it belongs to $\mathcal{A}_{p+1}(\mathbb{K})$). It is a non-empty union of consecutive intervals $[t_k, t_{k+1}]$. For all possible difficulties concerning either the use of admissible tuples, or the presence of connection matrices (which explains the $\varepsilon = \pm$ in (4) below, in case x is a knot t_k), we refer the reader to [1].

Theorem 2.1. *In the spline space $\mathbb{S}$ the following two statements are equivalent*:

(1) *blossoms exist in* $\mathbb{S}$;
(2) *the space* $\mathbb{E}$ *possesses ready-to-blossom bases relative to any n-tuple* $(x_1, \ldots, x_n) \in \mathcal{A}_n(\mathbb{K})$.

Assume that blossoms exist in $\mathbb{E}$*. Then, in any ready-to-blossom basis relative to any given* $(x_1,\dots,x_n)\in\mathbb{A}_n(\mathbb{K})$*, the first coordinate of any* $F\in\mathbb{E}$ *is the value* $f(x_1,\dots,x_n)$ *of its blossom* f *at* $(x_1,\dots,x_n)$*. Furthermore, for any pairwise distinct* $a_1,\dots,a_r\in I$ *and any positive integers* $\mu_1,\dots,\mu_r$ *such that* $p:=\sum_{i=1}^r\mu_i<n$*, if the* p*-tuple* $(x_1,\dots,x_p):=(a_1^{[\mu_1]},\dots,a_r^{[\mu_r]})$ *is admissible, we have*

$$\begin{aligned}&W\big(\Psi_n^{a_1\prime},\dots,\Psi_{n-\mu_1+1}^{a_1\prime},\dots,\Psi_n^{a_r\prime},\dots,\Psi_{n-\mu_r+1}^{a_r\prime}\big)\big(x^\varepsilon\big)\neq 0,\\ &W\big(\Psi_n^{a_1},\dots,\Psi_{n-\mu_1+1}^{a_1},\dots,\Psi_n^{a_r},\dots,\Psi_{n-\mu_r+1}^{a_r}\big)\big(x^\varepsilon\big)\neq 0\quad \textit{for all } x\in\mathcal{J}(x_1,\dots,x_p)\setminus\{a_1,\dots,a_r\}.\end{aligned}\tag{4}$$

Proof. Let $\mu_1,\dots,\mu_r$ be positive integers such that $p:=\sum_{i=1}^r\mu_i\leqslant n$. Choose any pairwise distinct $a_1,\dots,a_r\in I$ such that the p-tuple $(x_1,\dots,x_p):=(a_1^{[\mu_1]},\dots,a_r^{[\mu_r]})$ is admissible. For simplicity we denote by $(\Phi_1,\dots,\Phi_p)$ the sequence $(\Psi_n^{a_1},\dots,\Psi_{n-\mu_r+1}^{a_r})$. Assume that $\mathbb{1},\Phi_1,\dots,\Phi_p$ are linearly independent. Observe that this does hold in case $r=1$ by application of Theorem 1.2 in some space $\mathbb{E}_k$. Select any $(n-p)$ functions $\Phi_{p+1},\dots,\Phi_n\in\mathbb{E}$ so as to obtain a mother-function $\Phi=(\Phi_1,\dots,\Phi_p,\Phi_{p+1},\dots,\Phi_n)$ in the space $\mathbb{E}$. This choice makes it easy to check that

$$\Delta(x_1,\dots,x_p):=\bigcap_{i=1}^r \operatorname{Osc}_{n-m_i}\Phi(a_i)=\big\{X=(X_1,\dots,X_n)\in\mathbb{R}^n\ \big|\ X_1=\dots=X_p=0\big\}.$$

Suppose for a while that $p=n$. Then, $\Delta(x_1,\dots,x_n)$ reduces to the origin of $\mathbb{R}^n$. This proves that (ii) $\Rightarrow$ (i). Moreover, via affine maps, this also proves our assertion about the first coordinate of any $F\in\mathbb{E}$ in the ready-to-blossom basis $(\mathbb{1},\Phi_1,\dots,\Phi_n)$.

From now on, assume that $p<n$ and that blossoms exist in $\mathbb{S}$. Once and for all, we select an integer j such that $[t_j,t_{j+1}]\subset\mathcal{J}(x_1,\dots,x_p)$ and a non-trivial interval $\widehat{I}$ contained in $[t_j,t_{j+1}]\setminus\{a_1,\dots,a_r\}$. Choose any $x\in\widehat{I}$. Existence of blossoms in $\mathbb{S}$ implies that the point $\varphi(x_1,\dots,x_p,x^{[n-p]})$ is well defined. It is the only point $X=(X_1,\dots,X_n)\in\mathbb{R}^n$ which belongs both to $\operatorname{Osc}_p\Phi(x)$ and to $\Delta(x_1,\dots,x_p)$. Therefore, we have $X=\Phi(x)+\sum_{i=1}^p\lambda_i(x)\Phi^{(i)}(x)$, where the coefficients $\lambda_1(x),\dots,\lambda_p(x)$ are uniquely determined by the p equations

$$\sum_{i=1}^p\lambda_i(x)\Phi_j^{(i)}(x)=-\Phi_j(x),\quad j=1,\dots,p.\tag{5}$$

Note that if ever $x=t_j$ (resp. $x=t_{j+1}$), x is to be interpreted as t_j^+ (resp. t_{j+1}^-) everywhere in each left-hand side of (5). Subsequently, whenever necessary we shall adopt a similar convention. The system (5) being unisolvent, we have $W(\Phi_1',\dots,\Phi_p')(x)\neq 0$. Accordingly, for $0\leqslant s\leqslant n-p-1$, the function $y\in\widehat{I}\mapsto W(\mathbb{1},\Phi_1,\dots,\Phi_p,\Psi_{j+p+1}^x)(y)$ vanishes exactly j times at x (see Lemma 24 of [2]). The latter $(n-p)$ functions are thus linearly independent. The functions $\mathbb{1},\Phi_1,\dots,\Phi_p$ being linearly independent, it readily follows that so are the $(n+1)$ functions $\mathbb{1},\Phi_1,\dots,\Phi_p,\Psi_n^x,\dots,\Psi_{p+1}^x$. We thus have proved (ii) by induction on the number of pairwise distinct points involved, along with the first part of (4).

Let us now prove the second part of (4). It does hold for $r=1$ by application of (2) in some EC-space $\mathbb{E}_k$. With the same data, notations, and convention as above, assume that $W(\Phi_1,\dots,\Phi_p)(x)\neq 0$ for all $x\in\widehat{I}$, with $p\leqslant n-2$ (otherwise we have finished). This guarantees that the C^{n-p} function $\lambda_p:\widehat{I}\to\mathbb{R}$ never vanishes. Consider the function

$$\widehat{\Phi}(x):=\big(0,\dots,0,\widehat{\Phi}_{p+1},\dots,\widehat{\Phi}_n\big):=\varphi\big(x_1,\dots,x_p,x^{[n-p]}\big),\quad x\in\widehat{I}.$$

By repeated differentiation of the equality $\widehat{\Phi}(x)=\Phi(x)+\sum_{i=1}^p\lambda_i(x)\Phi^{(i)}(x)$, one can thus deduce that, for any $x\in\widehat{I}$, the vectors $\widehat{\Phi}'(x),\dots,\widehat{\Phi}^{(n-p)}(x)$ are linearly independent and that, for $0\leqslant q\leqslant n-p$, the (therefore q-dimensional) osculating flat $\operatorname{Osc}_q\widehat{\Phi}(x)$ is contained in $\operatorname{Osc}_{q+p}\widehat{\Phi}(x)\cap\Delta(x_1,\dots,x_p)$. From (5) one can check that the intersection $\operatorname{Osc}_{q+p}\widehat{\Phi}(x)\cap\Delta(x_1,\dots,x_p)$ is q-dimensional too. Accordingly,

$$\operatorname{Osc}_q\widehat{\Phi}(x)=\operatorname{Osc}_{q+p}\widehat{\Phi}(x)\cap\Delta(x_1,\dots,x_p)\quad\text{for all } x\in\widehat{I},\ 0\leqslant q\leqslant n-p.\tag{6}$$

By standard arguments, one can then derive that, for any pairwise distinct $b_1,\dots,b_s\in\widehat{I}$ and any positive integers $\nu_1,\dots,\nu_s$ summing to $n-p$, the s osculating flats $\operatorname{Osc}_{n-p-\nu_i}\widehat{\Phi}(b_i)$ have in common the only point

$\varphi(a_1^{[\mu_1]},\dots,a_r^{[\mu_r]},b_1^{[\nu_1]},\dots,b_s^{[\nu_s]})$. Let $\widehat{\mathbb{E}}$ be the linear space spanned on $\widehat{I}$ by all functions $\widehat{F}(x) := f(x_1,\dots,x_p,x^{[n-p]})$, F ranging over $\mathbb{E}$. For any $F \in \mathbb{E}$, we know that $\widehat{F}(x)$ is the coordinate of F in any first ready-to-blossom basis relative to $(x_1,\dots,x_p,x^{[n-p]})$. This implies that, for any $x \in \widehat{I}$,

$$\widehat{F}(x) = W(F,\Phi_1,\dots,\Phi_p)(x)/W(\mathbb{1},\Phi_1,\dots,\Phi_p)(x) = W(F,\Phi_1,\dots,\Phi_p)(x)/W\big(\Phi_1',\dots,\Phi_p'\big)(x). \tag{7}$$

From the statements above, we know that $\widehat{\mathbb{E}}$ is an $(n-p+1)$-dimensional W-space on $\widehat{I}$ in which blossoms exist. Select $a_{r+1} \in \widehat{I}$. By application of (7) and Lemma 24 of [2], for $p+1 \leqslant k \leqslant n$, the function $\Psi_k^{a_{r+1}}$ vanishes exactly $k-p$ times at a_{r+1}. Applying (2) in the EC-space $\widehat{\mathbb{E}}$ ensures that $W(\widehat{\Psi}_n^{a_{r+1}},\dots,\widehat{\Psi}_k^{a_{r+1}})$ does not vanish on $\widehat{I}\setminus\{a_{r+1}\}$. So does $W(\overline{\Psi}_n^{a_{r+1}},\dots,\overline{\Psi}_k^{a_{r+1}})$, where $\overline{\Psi}_k := W(\Phi_1,\dots,\Phi_p,\Psi_k^{a_{r+1}})$. Lemma 20 of [2] tells us that

$$W\big(\overline{\Psi}_n^{a_{r+1}},\dots,\overline{\Psi}_k^{a_{r+1}}\big)(x) = \big[W(\Phi_1,\dots,\Phi_p)(x)\big]^{n-k}\, W\big(\Phi_1,\dots,\Phi_p,\Psi_n^{a_{r+1}},\dots,\Psi_k^{a_{r+1}}\big)(x). \tag{8}$$

This shows that $W(\Phi_1,\dots,\Phi_p,\Psi_n^{a_{r+1}},\dots,\Psi_k^{a_{r+1}})$ never vanishes on $\widehat{I}\setminus\{a_{r+1}\}$. The proof is actually complete, due to the following remark. Given $r \geqslant 1$, and $a_1 < a_2 < \cdots < a_r < a_{r+1}$ in I, if $(a_1^{[\mu_1]},\dots,a_{r+1}^{[\mu_{r+1}]})$ is admissible, then both $(a_1^{[\mu_1]},\dots,a_r^{[\mu_r]})$ and $(a_2^{[\mu_2]},\dots,a_{r+1}^{[\mu_{r+1}]})$ are admissible and $\mathcal{J}(a_1^{[\mu_1]},\dots,a_{r+1}^{[\mu_{r+1}]})$ is the union of $\mathcal{J}(a_1^{[\mu_1]},\dots,a_r^{[\mu_r]})$ and $\mathcal{J}(a_2^{[\mu_2]},\dots,a_{r+1}^{[\mu_{r+1}]})$. □

3. Proof of Theorem 1.4

We shall just give a condensed proof of Theorem 3.1 below via ready-to-blossom bases and under the least possible differentiability assumption. It is at the root of all design algorithms for splines, *e.g.*, the de Boor evaluation algorithm which naturally produces B-spline bases. It is also the major difficulty in the proof of Theorem 1.4, the remaining part being similar to that in [1] which we refer the reader to.

Theorem 3.1. *Assume that blossoms exist in* $\mathbb{S}$*. Then, they are pseudoaffine in each variable, in the sense that, for any* $(x_1,\dots,x_{n-1}) \in \mathcal{A}_{n-1}(\mathbb{K})$*, the function* $\varphi(x_1,\dots,x_{n-1},.)$ *– which takes its values in an affine line – is strictly monotone on any interval* J *such that* $x,y \in J \Rightarrow (x_1,\dots,x_{n-1},x,y) \in \mathcal{A}_{n+1}(\mathbb{K})$.

Proof. If $(x_1,\dots,x_{n-1}) = (a_1^{[\mu_1]},\dots,a_r^{[\mu_r]})$ (positive μ_i, pairwise distinct a_i), it is sufficient to prove the strict monotonicity on the largest interval $J_1 \subset J\setminus\{a_2,\dots,a_r\}$ which contains a_1. Choose a ready-to-blossom basis $(\mathbb{1},\Phi_1,\dots,\Phi_n)$ relative to $(x_1,\dots,x_{n-1},a_1)$, and consider the associated mother-function $\Phi = (\Phi_1,\dots,\Phi_n)$. It is sufficient to prove that $\overline{\varphi}_n := \varphi_n(x_1,\dots,x_{n-1},.)$ is C^1 and strictly monotone on J_1, with therefore $\Phi_n := \Psi_{n-\mu_1}^{a_1}$. From (7) and (4) we can see that $\overline{\varphi}_n$ is C^1 on $J_1\setminus\{a_1\}$. Moreover, Lemma 17 of [2] ensures that, for any $x \in J_1\setminus\{a_1\}$,

$$\overline{\varphi}_n'(x) = (-1)^{n-1} W(\Phi_1,\dots,\Phi_{n-1})(x) W\big(\Phi_1',\dots,\Phi_{n-1}',\Phi_n'\big)(x)/\big[W\big(\Phi_1',\dots,\Phi_{n-1}'\big)(x)\big]^2.$$

Accordingly, (4) shows that $\overline{\varphi}_n'(x) \neq 0$ on $J_1\setminus\{a_1\}$. Lemma 31 of [2] enables us to say how each of the latter Wronskians behave close to a_1. This eventually shows that $\overline{\varphi}_n'(x) \to C_\varepsilon$ when $x \to a_1^\varepsilon$ in $J_1\setminus\{a_1\}$, for some non-zero $C_\varepsilon \in \mathbb{R}$. Suppose that a_1 is interior to J_1. If a_1 is not a knot, then $C_- = C_+$. Otherwise, use Lemma 49 of [2] to show that C_- and C_+ have the same strict sign. One can similarly prove that $\overline{\varphi}_n(x) \sim C_\varepsilon(x-a_1)$ for $x \in J_1\setminus\{a_1\}$ close to a_1. The proof is complete. □

References

[1] M.-L. Mazure, On the equivalence between existence of B-spline bases and existence of blossoms, Constructive Approximation 20 (2004) 603–624.

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