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Partial Differential Equations/Functional Analysis

Geometry of Sobolev spaces with variable exponent: smoothness and uniform convexity

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Abstract

Let $\Omega \subset \mathbf{R}^N$, $N \geqslant 2$, be a smooth bounded domain. It is shown that: (a) if $p \in L^\infty(\Omega)$ and $\operatorname{ess\,inf}_{x\in\Omega} p(x) > 1$, then the generalized Lebesgue space $(L^{p(\cdot)}(\Omega), \| \, \|_{p(\cdot)})$ is smooth; (b) if $p \in \mathcal{C}(\bar{\Omega})$ and $p(x) > 1$, for all $x \in \bar{\Omega}$, then the generalized Sobolev space $(W_0^{1,p(\cdot)}(\Omega), \| \, \|_{1,p(\cdot)})$ is smooth. In both situations, the formulae giving the Gâteaux derivative of the norm, corresponding to each of the above spaces, are given; (c) if $p \in \mathcal{C}(\bar{\Omega})$ and $p(x) \geqslant 2$, for all $x \in \bar{\Omega}$, then $(W_0^{1,p(\cdot)}(\Omega), \| \, \|_{1,p(\cdot)})$ is uniformly convex and smooth. ***To cite this article: G. Dinca, P. Matei, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Géométrie des espaces de Sobolev à coefficients variables : lissitude et convexité uniforme. Soit $\Omega \subset \mathbf{R}^N$, $N \geqslant 2$, un domain borné et régulier. On demontre que : (a) si $p \in L^\infty(\Omega)$ et $\operatorname{ess\,inf}_{x\in\Omega} p(x) > 1$, alors l'espace de Lebesgue généralisé $(L^{p(\cdot)}(\Omega), \| \, \|_{p(\cdot)})$ est lisse ; (b) si $p \in \mathcal{C}(\bar{\Omega})$ et $p(x) > 1$, pour tout $x \in \bar{\Omega}$, alors l'espace de Sobolev généralisé $(W_0^{1,p(\cdot)}(\Omega), \| \, \|_{1,p(\cdot)})$ est lisse. Dans les deux cas, les formules de la dérivée au sens de Gâteaux de chaque norme des espaces ci-dessus sont données ; (c) si $p \in \mathcal{C}(\bar{\Omega})$ et $p(x) \geqslant 2$, pour tout $x \in \bar{\Omega}$, alors $(W_0^{1,p(\cdot)}(\Omega), \| \, \|_{1,p(\cdot)})$ est uniformément convexe et lisse. ***Pour citer cet article : G. Dinca, P. Matei, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Version française abrégée

Le résultat principal de cette Note est le théorème suivant :

Théorème 1. *Soit $\Omega \subset \mathbf{R}^N$, $N \geqslant 2$, un domaine borné et régulier.*
(a) *Si $p \in L_+^\infty(\Omega)$ et $p^- > 1$, alors $(L^{p(\cdot)}(\Omega), \| \, \|_{p(\cdot)})$ est lisse. En tout $u \in L^{p(\cdot)}(\Omega) \setminus \{0\}$, la dérivée au sens de Gâteaux de la norme $\| \, \|_{p(\cdot)}$ est donnée par :*

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$$\big\langle \|\ \|'_{p(\cdot)}(u), h\big\rangle = \frac{\int_\Omega p(x)\frac{|u(x)|^{p(x)-1}\operatorname{sgn} u(x)}{\|u\|_{p(\cdot)}^{p(x)}} h(x)\,\mathrm{d}x}{\int_\Omega p(x)\frac{|u(x)|^{p(x)}}{\|u\|_{p(\cdot)}^{p(x)+1}}\,\mathrm{d}x}, \quad \textit{pour tout } h \in L^{p(\cdot)}(\Omega).$$

(b) *Si* $p \in \mathcal{C}(\bar{\Omega})$ *et* $p(x) > 1$, *pour tout* $x \in \bar{\Omega}$, *alors* $(W_0^{1,p(\cdot)}(\Omega), \|\ \|_{1,p(\cdot)})$ *est lisse. En tout* $u \in W_0^{1,p(\cdot)}(\Omega) \setminus \{0\}$, *la dérivée au sens de Gâteaux de la norme* $\|\ \|_{1,p(\cdot)}$ *est donnée par :*

$$\big\langle \|\ \|'_{1,p(\cdot)}(u), h\big\rangle = \frac{\int_\Omega p(x)\frac{|\nabla u(x)|^{p(x)-2}\nabla u \nabla h}{\|u\|_{1,p(\cdot)}^{p(x)}}\,\mathrm{d}x}{\int_\Omega p(x)\frac{|\nabla u|^{p(x)}}{\|u\|_{1,p(\cdot)}^{p(x)+1}}\,\mathrm{d}x}, \quad \textit{pour tout } h \in W_0^{1,p(\cdot)}(\Omega).$$

(c) *Si* $p \in \mathcal{C}(\bar{\Omega})$ *et* $p(x) \geqslant 2$, *pour tout* $x \in \bar{\Omega}$, *alors* $(W_0^{1,p(\cdot)}(\Omega), \|\ \|_{1,p(\cdot)})$ *est uniformément convexe et lisse.*

Les définitions des principales notions intervenant dans cet enoncé ainsi que les grandes lignes de la démonstration sont données dans la version anglaise ci-dessous. Pour les détails, voir [2].

1. Basic definitions

Let $\Omega \subset \mathbf{R}^N$, $N \geqslant 2$, be a bounded and smooth domain. We take Lebesgue measure in $\mathbf{R}^N$ and denote by

$$L_+^\infty(\Omega) = \Big\{u \in L^\infty(\Omega) \;\Big|\; 1 \leqslant p^- = \operatorname*{ess\,inf}_\Omega u \leqslant p^+ = \operatorname*{ess\,sup}_\Omega u < \infty\Big\}.$$

For $p \in L_+^\infty(\Omega)$, the generalized Lebesgue space $L^{p(\cdot)}(\Omega)$ is defined by

$$L^{p(\cdot)}(\Omega) = \Big\{u : \Omega \to \mathbf{R} \;\Big|\; u \text{ measurable and } \rho_{p(\cdot)}(u) = \int_\Omega \big|u(x)\big|^{p(x)}\,\mathrm{d}x < \infty\Big\}$$

and it is endowed with the norm

$$\|u\|_{p(\cdot)} = \inf\Big\{\lambda > 0 \;\Big|\; \rho_{p(\cdot)}\Big(\frac{u}{\lambda}\Big) = \int_\Omega \Big|\frac{u(x)}{\lambda}\Big|^{p(x)}\,\mathrm{d}x \leqslant 1\Big\}, \quad \text{for all } u \in L^{p(\cdot)}(\Omega).$$

The generalized Sobolev space $W^{1,p(\cdot)}(\Omega)$ is defined by

$$W^{1,p(\cdot)}(\Omega) = \big\{u \in L^{p(\cdot)}(\Omega) \;\big|\; |\nabla u| \in L^{p(\cdot)}(\Omega)\big\}, \quad |\nabla u|^2 = \sum_{i=1}^N \Big(\frac{\partial u}{\partial x_i}\Big)^2$$

and it is endowed with the norm

$$\|u\| = \|u\|_{p(\cdot)} + \big\||\nabla u|\big\|_{p(\cdot)}, \quad \text{for all } u \in W^{1,p(\cdot)}(\Omega).$$

We define $W_0^{1,p(\cdot)}(\Omega)$ as the closure of $\mathcal{C}_0^\infty(\Omega)$ in $(W^{1,p(\cdot)}(\Omega), \|\ \|)$. If $p \in \mathcal{C}(\bar{\Omega})$ and $p(x) > 1$ for any $x \in \bar{\Omega}$, there is a constant $c > 0$ such that (*Poincaré's inequality*)

$$\|u\|_{p(\cdot)} \leqslant c\big\||\nabla u|\big\|_{p(\cdot)}, \quad \text{for all } u \in W_0^{1,p(\cdot)}(\Omega)$$

holds and, from this, we infer that $\|u\|$ and $\|u\|_{1,p(\cdot)} = \||\nabla u|\|_{p(\cdot)}$ are equivalent norms on $W_0^{1,p(\cdot)}(\Omega)$.

For the basic properties of the above spaces, cf. Edmunds and Rákosník [3,4], Fan and Zhao [5], Kováčik and Rákosník [7]. In what follows, $W_0^{1,p(\cdot)}(\Omega)$ will be considered as endowed with the norm $\|\ \|_{1,p(\cdot)}$ and we will often write $W_0^{1,p(\cdot)}(\Omega)$ instead of $(W_0^{1,p(\cdot)}(\Omega), \|\ \|_{1,p(\cdot)})$.

2. The main result

Theorem 1. *Let $\Omega \subset \mathbf{R}^N$, $N \geqslant 2$, be a bounded and smooth domain.*

(a) *If $p \in L^{\infty}_{+}(\Omega)$ and $p^- > 1$, then $(L^{p(\cdot)}(\Omega), \| \ \|_{p(\cdot)})$ is smooth.*
At any $u \in L^{p(\cdot)}(\Omega)$, $u \neq 0$, the Gâteaux derivative of the norm $\| \ \|_{p(\cdot)}$ is given by

$$\left\langle \| \ \|'_{p(\cdot)}(u), h \right\rangle = \frac{\int_\Omega p(x) \frac{|u(x)|^{p(x)-1} \operatorname{sgn} u(x)}{\|u\|_{p(\cdot)}^{p(x)}} h(x)\, \mathrm{d}x}{\int_\Omega p(x) \frac{|u(x)|^{p(x)}}{\|u\|_{p(\cdot)}^{p(x)+1}}\, \mathrm{d}x}, \quad \text{for all } h \in L^{p(\cdot)}(\Omega). \tag{1}$$

(b) *If $p \in \mathcal{C}(\bar{\Omega})$ and $p(x) > 1$, for all $x \in \bar{\Omega}$, then $(W_0^{1,p(\cdot)}(\Omega), \| \ \|_{p(\cdot)})$ is smooth.*
At any $u \in W_0^{1,p(\cdot)}(\Omega)$, $u \neq 0$, the Gâteaux derivative of the norm $\| \ \|_{1,p(\cdot)}$ is given by

$$\left\langle \| \ \|'_{1,p(\cdot)}(u), h \right\rangle = \frac{\int_\Omega p(x) \frac{|\nabla u(x)|^{p(x)-2} \nabla u \nabla h}{\|u\|_{1,p(\cdot)}^{p(x)}}\, \mathrm{d}x}{\int_\Omega p(x) \frac{|\nabla u|^{p(x)}}{\|u\|_{1,p(\cdot)}^{p(x)+1}}\, \mathrm{d}x}, \quad \text{for all } h \in W_0^{1,p(\cdot)}(\Omega). \tag{2}$$

(c) *If $p \in \mathcal{C}(\bar{\Omega})$ and $p(x) \geqslant 2$, for all $x \in \bar{\Omega}$, then $(W_0^{1,p(\cdot)}(\Omega), \| \ \|_{p(\cdot)})$ is uniformly convex and smooth.*

Next, we give the idea of the proof (for more details, see [2]).

(a) The key tools in proving (a) is the classical implicit function theorem and a basic property of the convex modular $\rho_{p(\cdot)}$.
Indeed, we have to prove that, for a given $u_0 \in L^{p(\cdot)}(\Omega)$, $u_0 \neq 0$, and any $h \in L^{p(\cdot)}(\Omega)$, the function $t \in \mathbf{R}$, $t \to \|u_0 + th\|_{p(\cdot)}$ is differentiable at $t = 0$. Let $k > 1$ be a fixed real number, $D = (1, 1) \times (\frac{1}{k}\|u_0\|_{p(\cdot)}, k\|u_0\|_{p(\cdot)})$ and $\Phi : D \to \mathbf{R}$ defined by

$$\Phi(t, \lambda) = \rho_{p(\cdot)}\left(\frac{u_0 + th}{\lambda}\right) - 1 = \int_\Omega \frac{|u_0(x) + th(x)|^{p(x)}}{\lambda^{p(x)}}\, \mathrm{d}x - 1. \tag{3}$$

It may be shown that:
(a) $\Phi \in \mathcal{C}^1(D)$;
(b) $\Phi(0, \|u_0\|_{p(\cdot)}) = 0$;
(c) $\frac{\partial \Phi}{\partial \lambda}(0, \|u_0\|_{p(\cdot)}) < 0$.
Consequently, there exist neighborhoods U of 0 and V of $\|u_0\|_{p(\cdot)}$ such that $U \times V \subset D$ and a $\mathcal{C}^1$-mapping $\lambda : U \to V$ which satisfies: $\lambda(0) = \|u_0\|_{p(\cdot)}$, $\Phi(t, \lambda(t)) = 0$ for any $t \in U$ and

$$\lambda'(t) = -\frac{\frac{\partial \Phi}{\partial t}(t, \lambda(t))}{\frac{\partial \Phi}{\partial \lambda}(t, \lambda(t))}, \quad \text{for all } t \in U. \tag{4}$$

Taking into account the definition of Φ (see (3)), $\Phi(t, \lambda(t)) = 0$ for any $t \in U$ rewrites as

$$\rho_{p(\cdot)}\left(\frac{u_0 + th}{\lambda(t)}\right) = 1, \quad \text{for all } t \in U. \tag{5}$$

Since $\|u\|_{p(\cdot)} = a > 0$ if and only if $\rho_{p(\cdot)}(\frac{u}{a}) = 1$ (Fan and Zhao [5, Theorem 1.2]) we deduce from (5) that

$$\lambda(t) = \|u_0 + th\|_{p(\cdot)}, \quad \text{for all } t \in U. \tag{6}$$

By combining (4) and (6) we derive, in particular, that $\lambda'(0)$ exists and

$$\lambda'(0) = \lim_{t \to 0} \frac{\|u_0 + th\|_{p(\cdot)} - \|u_0\|_{p(\cdot)}}{t} = -\frac{\frac{\partial \Phi}{\partial t}(0, \|u_0\|_{p(\cdot)})}{\frac{\partial \Phi}{\partial \lambda}(0, \|u_0\|_{p(\cdot)})}, \tag{7}$$

i.e. the norm $\| \ \|_{p(\cdot)}$ is Gâteaux differentiable at u_0 and the Gâteaux derivative of this norm at u_0 is given by

$$\langle \| \ \|'_{p(\cdot)}(u_0), h\rangle = -\frac{\frac{\partial \Phi}{\partial t}(0, \|u_0\|_{p(\cdot)})}{\frac{\partial \Phi}{\partial \lambda}(0, \|u_0\|_{p(\cdot)})}, \quad \text{for all } h \in W_0^{1,p(\cdot)}(\Omega), \tag{8}$$

with Φ given by (3).

Since a Banach space is smooth if and only if its norm is Gâteaux differentiable (Diestel [1, chapter 2, §1, Theorem 1]) we conclude on the smoothness of $(L^{p(\cdot)}(\Omega), \| \ \|_{p(\cdot)})$.

In order to obtain formula (1) it is enough to compute explicitly the derivatives $\frac{\partial \Phi}{\partial t}(0, \|u_0\|_{p(\cdot)})$ and $\frac{\partial \Phi}{\partial \lambda}(0, \|u_0\|_{p(\cdot)})$ in (8).

The key point in so doing is to show that these derivatives may be computed by "differentiating under the integral sign", i.e. at any $(t, \lambda) \in D$, $\frac{\partial \Phi}{\partial t}(t, \lambda) = \int_\Omega [\frac{\partial}{\partial t}(\frac{|u_0(x)+th(x)|^{p(x)}}{\lambda^{p(x)}})]\,dx$ and a similar formula for $\frac{\partial \Phi}{\partial \lambda}(t, \lambda)$. In these computations the condition $p^- > 1$ is essentially involved (see [2] for necessary details).

(b) Since at any $u \in W_0^{1,p(\cdot)}(\Omega)$, $u \neq 0_{W_0^{1,p(\cdot)}(\Omega)}$ ($\Leftrightarrow |\nabla u| \neq 0_{L^{p(\cdot)}(\Omega)}$) the functional $\Psi(u) = \|u\|_{1,p(\cdot)} = \||\nabla u|\|_{p(\cdot)}$ may be written as a composed function $\Psi(u) = Q(Pu)$ with

$$P : W_0^{1,p(\cdot)}(\Omega) \to L^{p(\cdot)}(\Omega), \quad Pu = |\nabla u|,$$

$$Q : L^{p(\cdot)}(\Omega) \to \mathbf{R}, \quad Qv = \|v\|_{p(\cdot)},$$

the differentiation chain rule gives us

$$\langle \| \ \|'_{1,p(\cdot)}(u), h\rangle = \langle Q'(Pu), P'(u)h\rangle = \langle \| \ \|'_{p(\cdot)}(|\nabla u|), P'(u)h\rangle, \quad \text{for all } h \in W_0^{1,p(\cdot)}(\Omega).$$

Thus, by simple computation and using (1), formula (2) follows.

The following auxiliary results are needed for the proof of point (c):

Proposition 1. *Let $A(t) = \int_0^{|t|} a(s)\,ds$, $t \in \mathbf{R}$ be an N function (cf. [8], p. 6) with $\frac{a(s)}{s}$ non-decreasing on $(0, \infty)$. Let X be a real vector space and $T : X \times X \to \mathbf{R}$ a bilinear symmetric and positive form. Then, for any $f, g \in X$, one has*:

$$A\big(\sqrt{T(f,f)}\big) + A\big(\sqrt{T(g,g)}\big) \geqslant 2\left[A\left(\sqrt{T\left(\frac{f+g}{2}, \frac{f+g}{2}\right)}\right) + A\left(\sqrt{T\left(\frac{f-g}{2}, \frac{f-g}{2}\right)}\right)\right]. \tag{9}$$

This result is due to Gröger [6] (see also Langenbach [9, p. 153] and [2] for a simple proof).

Proposition 2. *Let $p \in L_+^\infty(\Omega)$. Then one has*:

(a) *For every $\varepsilon > 0$ there exists a number $\eta(\varepsilon) > 0$ such that if $u \in L^{p(\cdot)}(\Omega)$ satisfies $\|u\|_{p(\cdot)} \geqslant \varepsilon$, then $\rho_{p(\cdot)}(u) \geqslant \eta$*;

(b) *For every $\eta \in (0, 1)$ there exists a number $\delta \in (0, 1)$ such that if $u \in L^{p(\cdot)}(\Omega) \setminus \{0\}$ satisfies $\rho_{p(\cdot)}(u) \leqslant 1 - \eta$, then $\|u\|_{p(\cdot)} \leqslant 1 - \delta$.*

Proof of point (b). (Point (a) may be proven by using the same type of arguments.)

Assuming the contrary, a number $\eta \in (0, 1)$ and a sequence $(u_n) \subset L^{p(\cdot)}(\Omega) \setminus \{0\}$ such that $\rho_{p(\cdot)}(u_n) \leqslant 1 - \eta$ and $\|u_n\| > 1 - \frac{1}{n}$ would exist.

From $\rho_{p(\cdot)}(u_n) \leqslant 1 - \eta$ we derive (see [5, Theorem 1.3]) that $\|u_n\|_{p(\cdot)} < 1$.

Thus, $\|u_n\| \to 1$ as $n \to \infty$.

Since ([5, Theorem 1.3]) $\|u_n\|_{p(\cdot)}^{p^+} \leqslant \rho_{p(\cdot)}(u_n) \leqslant \|u_n\|_{p(\cdot)}^{p^-}$, it follows that $\rho_{p(\cdot)}(u_n) \to 1$, in contradiction with $\rho_{p(\cdot)}(u_n) \leqslant 1 - \eta$. □

Proof of point (c) in Theorem 1. Let $x \in \Omega$. Consider the N-function $A(t) = |t|^{p(x)}$. Since $a(s) = p(x)s^{p(x)-1}$ and $p(x) \geqslant 2$, it follows that $\frac{a(s)}{s}$ is non-decreasing in $(0, \infty)$. Let $T : W_0^{1,p(\cdot)}(\Omega) \times W_0^{1,p(\cdot)}(\Omega) \to \mathbf{R}$ be the bilinear form defined by $T(f, g) = (\nabla f)(x) \cdot (\nabla g)(x)$. Inequality (9) applies and we obtain:

$$\left|\nabla f(x)\right|^{p(x)} + \left|\nabla g(x)\right|^{p(x)} \geqslant 2\left[\left|\nabla\left(\frac{f+g}{2}\right)(x)\right|^{p(x)} + \left|\nabla\left(\frac{f-g}{2}\right)(x)\right|^{p(x)}\right].$$

By integrating over Ω we get:

$$\rho_{p(\cdot)}\big(|\nabla f|\big) + \rho_{p(\cdot)}\big(|\nabla g|\big) \geqslant 2\left[\rho_{p(\cdot)}\left(\left|\nabla\left(\frac{f+g}{2}\right)\right|\right) + \rho_{p(\cdot)}\left(\left|\nabla\left(\frac{f-g}{2}\right)\right|\right)\right]. \tag{10}$$

Now, let $\varepsilon \in (0, 2]$ be given and let $f, g \in W_0^{1,p(\cdot)}(\Omega)$ be such that

$$\|f\|_{1,p(\cdot)} = \big\||\nabla f|\big\|_{p(\cdot)} = 1, \tag{11}$$

$$\|g\|_{1,p(\cdot)} = \big\||\nabla g|\big\|_{p(\cdot)} = 1, \tag{12}$$

$$\|f-g\|_{1,p(\cdot)} = \big\|\left|\nabla(f-g)\right|\big\|_{p(\cdot)} \geqslant \varepsilon \Leftrightarrow \left\|\left|\nabla\left(\frac{f-g}{2}\right)\right|\right\| \geqslant \frac{\varepsilon}{2}. \tag{13}$$

From (11) and (12) we deduce (cf. [5, Theorem 1.2]) that $\rho_{p(\cdot)}(|\nabla f|) = 1$, $\rho_{p(\cdot)}(|\nabla g|) = 1$ while from (13) and Proposition 2(a) it follows that there exists $\eta > 0$ such that $\rho_{p(\cdot)}(|\nabla(\frac{f-g}{2})|) \geqslant \eta$ (clearly, we may assume $\eta < 1$).

Consequently, from (10) it follows that $\rho_{p(\cdot)}(|\nabla(\frac{f+g}{2})|) \leqslant 1 - \eta$ and from Proposition 2(b) we deduce the existence of a number $\delta \in (0, 1)$ such that $\||\nabla(\frac{f+g}{2})|\|_{p(\cdot)} \leqslant 1 - \delta$ which rewrites as $\|\frac{f-g}{2}\|_{1,p(\cdot)} \leqslant 1 - \delta$. □

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