Dynamical Systems/Probability Theory

Some optimal pointwise ergodic theorems with rate

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Abstract

Let $T$ be a Dunford–Schwartz operator on the probability space $(X, \Sigma, \mu)$ and $p > 1$. For $f$ in the range of suitable operators of $L^p(X, \Sigma, \mu)$, we obtain pointwise ergodic theorems with rate, using a method of Derriennic and Lin (2001). When $T$ is induced by a $\mu$-preserving transformation, these results are shown to be optimal. The proof of the optimality is inspired from a construction of Déniel (1989).

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Résumé

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Version française abrégée

Soit $T$ un opérateur de Dunford–Schwartz sur l’espace de probabilité $(X, \Sigma, \mu)$ (i.e. $T$ contracte chaque espace $L^r(X, \mu)$, $1 \leq r \leq \infty$). Pour $\alpha \in [0, 1]$, Derriennic et Lin [5] ont utilisé le développement en série entière $(1 - t)^\alpha = 1 - \sum_{n \geq 1} a_n t^n$, où $a_n = a_n(\alpha) > 0$ et $\sum_{n \geq 1} a_n = 1$, pour définir l’opérateur $(I - T)^\alpha$ sur $L^p(X, \mu)$, par

$$(I - T)^\alpha = I - \sum_{n \geq 1} a_n T^n.$$

Pour $f \in (I - T)^\alpha L^p(X, \mu)$, des théorèmes ergodiques ponctuels avec vitesse sont établis dans [5]. Cette approche a ensuite été développée par Zhao et Woodroofe [10], puis dans [3], en considérant des séries entières plus générales. Dans ces derniers travaux, le seul cas $p = 2$ a été traité, en raison des applications en vue. Dans cette note nous montrons que la méthode s’applique aisément à tout $p > 1$. Puis, nous montrons que les vitesses de convergence

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obtenues dans le théorème ergodique ponctuel sont, en un certain sens, optimales. Les résultats sont particulièrement précis dans le cas \( p = 2 \). La preuve de l’optimalité s’inspire d’une construction due à Déniel [4]. Par exemple, en corollaire nous obtenons

**Théorème.** Soit \((X, \Sigma, \mu, \theta)\) un système dynamique (non nécessairement ergodique). Si \( f \in L^2(X, \mu) \) satisfait

\[
\sum_{n \geq 2} \log n (\log \log n)^{1+\varepsilon} \frac{\|f \circ \theta^n + f \circ \theta^{n+1}\|_2^2}{n^r} < +\infty, \text{ pour } \varepsilon > 0, \text{ alors}
\]

\[
\frac{1}{n} \sum_{k=1}^{n} f \circ \theta^k \xrightarrow{n \to +\infty} 0 \quad \mu\text{-a.s.}
\]

Par contre, sur tout système dynamique ergodique, inversible et non atomique \((X, \Sigma, \mu, \theta)\), il existe \( f \in L^2(X, \mu) \) (centrée), satisfaisant la condition \( \sum_{n \geq 2} \log n \log \log n \frac{\|f \circ \theta^n + f \circ \theta^{n+1}\|_2^2}{n^r} < +\infty \), mais telle que \( \limsup_{n \to +\infty} \frac{1}{\sqrt{n}} \times |\sum_{k=1}^{n} f \circ \theta^k| = +\infty \) \( \mu\)-a.s.

1. **Introduction**

   Let \( T \) be a Dunford–Schwartz operator on the probability space \((X, \Sigma, \mu)\) (i.e. \( T \) is a contraction of each \( L^r(X, \mu) \), \( 1 \leq r \leq \infty \)). For \( 0 < \alpha < 1 \), Derriennic and Lin [5] used the power series expansion \((1 - t)^\alpha = 1 - \sum_{n \geq 1} a_n t^n\), where \( a_n = a_n(\alpha) > 0 \) and \( \sum_{n \geq 1} a_n = 1 \), to define the operator \((I - T)^\alpha\) on \( L^p(X, \mu) \) by

\[
(I - T)^\alpha = I - \sum_{n \geq 1} a_n T^n.
\]

For \( p > 1 \) and \( f \in (I - T)^\alpha L^p(X, \mu) \), pointwise ergodic theorems with rate were obtained in [5]. In this note we use generalized power series of operators (as in Zhao–Woodroofe [10] or [3]) to obtain more precise rates. Moreover, inspired by a construction of Déniel [4], we show the optimality of the obtained rates.

2. **The results**

   Let \( b \) be a slowly varying function (as in Zygmund [11, p. 186]) and fix \( \alpha \in [0, 1[ \). We consider the series \( B(z) := \sum_{n \geq 1} \beta_n z^n \), where \( \beta_n = \frac{C}{n} \sum_{k \geq n} \frac{b(k)}{k^{1 + \alpha}} \) (notice that \( \sum_{k \geq n} \frac{b(k)}{k^{1 + \alpha}} \sim C \frac{\alpha n}{\alpha n^{\alpha}} \)), and \( C \) is such that \( \sum_{n \geq 1} \beta_n = 1 \). The series defining \( B(z) \) is absolutely convergent on \( \overline{D} \) \((D := \{ z \in \mathbb{C} : |z| < 1 \})\) and defines a continuous function on \( \overline{D} \), analytic on \( D \). Moreover, \( B(z) \) is a (strict) convex combination of complex numbers of \( \overline{D} \), hence \( B(z) = 1 \Leftrightarrow z = 1 \) and the function \( A(z) := 1/(1 - B(z)) \) is well defined, continuous on \( \overline{D} - \{ 1 \} \), analytic on \( D \). So, there exists \( \{a_n\}_{n \in \mathbb{N}} \subset \mathbb{C} \), such that

\[
A(z) = \sum_{n \geq 1} a_n z^n \quad \forall z \in D.
\]

For a power bounded operator \( T \) on a Banach space \( B \), the operator \( B(T) := \sum_{n \geq 1} \beta_n T^n \) is well-defined. Define also \( A_n(T) := \sum_{k=0}^{n} a_k T^k \), \( n \in \mathbb{N} \). For a given \( T \) we denote \( B = B(T) \) and \( A_n = A_n(T) \).

For \( f \in B \) such that \( \{A_n f\} \) converges in \( B \), we denote by \( A(f) \) its limit and say that \( A(f) \) converges. The following is proved in [2] or [5] for some special cases:

**Proposition 2.1.** Let \( T \) be a power bounded operator on a Banach space \( B \). If \( f \in B \) is such that \( A(f) \) converges in \( B \), then \( f \) and \( h := A(f) \) are in \((I - T)^\alpha B \) and satisfy \( f = (I - B)h \).

Conversely, if \( f \in (I - B)(I - T)^\alpha B \), then \( A(f) \) converges.

In the sequel, \( T \) will be a Dunford–Schwartz operator on \((X, \Sigma, \mu)\) and \( B \) will be \( L^p(X, \mu) \), with \( p > 1 \). In order to find conditions for the convergence of \( A(f) \) we need the following, proved in [3] for \( \alpha = 1/2 \):

**Proposition 2.2.** Let \( A \) be as above. Then \( a_n > 0, \forall n \geq 0 \), and there exist \( L_\alpha, K > 0 \), such that
Proposition 2.3. Let \( (X, \Sigma, \mu) \) be a probability space and \( T \) be a Dunford–Schwartz operator, whose restriction to \( L^2(X, \mu) \) is normal. Let \( b \) be a slowly varying function and \( \alpha \in [0, 1] \). Then \( f \in (I - B)L^p(X, \mu) \), whenever \( f \) satisfies the condition

\[
(C_{\rho}) \quad \sum_{n \geq 1} \frac{\| S_n(f) \|_p}{n^{2 - \alpha} b(n)^2} < +\infty.
\]

We recall that an operator \( T \) on \( L^2(X, \mu) \) is normal if \( T^*T = TT^* \). The next proposition follows from Lemma 2.1 and Proposition 2.3 of [3] (see also the proof of Theorem 3.3 there) using (ii) and (iii) of Proposition 2.2:

Proposition 2.4. Let \( (X, \Sigma, \mu) \) be a probability space and \( T \) be a Dunford–Schwartz operator, whose restriction to \( L^2(X, \mu) \) is normal. Let \( b \) be a slowly varying function and \( \alpha \in ]0, 1[ \). Then \( f \in (I - B)L^p(X, \mu) \) if and only if

\[
(C_{\rho}^2) \quad \sum_{n \geq 1} \frac{\| S_n(f) \|_p^2}{n^{2 - 2\alpha} b(n)^4} < +\infty.
\]

For \( p > 1 \), define the dual index \( q := p/(p - 1) \). Our main results are the following.

Theorem 2.5. Let \( T \) be a Dunford–Schwartz operator on \( (X, \Sigma, \mu) \). Let \( b \) be a slowly varying function, \( \alpha \in [0, 1[ \), \( p > 1 \), and \( B \) as above. Let \( f \in (I - B)L^p(X, \Sigma, \mu) \) (e.g. \( f \) satisfies \( (C_{\rho}) \)). Then

\[
\frac{\sum_{k=1}^{n} T^k f}{n^{1/p} \left( \sum_{k=1}^{n} \frac{b(k)^q}{b(n)^{q'}} \right)^{1/q}} \to 0 \quad \mu\text{-a.s.}
\]

Moreover, if \( (X, \Sigma, \mu) \) is non-atomic and \( T \) is induced by an ergodic invertible measure preserving transformation \( \theta \), then for every positive function \( \psi \) satisfying \( \lim_{n \to +\infty} \frac{\sum_{k=0}^{n-1} \frac{b(k)^q}{b(n)^{q'}}}{\psi(n)} = +\infty \), there exists \( f \in (I - B)L^p(X, \Sigma, \mu) \) (hence \( \int_X f \, d\mu = 0 \)) such that

\[
\limsup_{n \to +\infty} \frac{\left| \sum_{k=1}^{n} f \circ \theta^k \right|}{n^{1/p} \psi(n)} = +\infty \quad \mu\text{-a.s.}
\]

Remark 1. When \( b \equiv 1 \) we recover Theorem 3.2 of [5], hence Theorem 2.5 shows the optimality of Theorem 3.2 of [5] in the above sense. Weber [9] and Cohen–Lin [1] obtained pointwise ergodic theorems with rate, in the context
of power-bounded operators in $L^p(X, \mu)$. The use of condition $(C_p)$ in Theorem 2.6 yields in case (iii) a better rate than that of [9] or [1]; but, our rate is not as good. A similar discussion holds in case (ii) according to the chosen function $b$.

**Theorem 2.7.** Let $T$ be a Dunford–Schwartz operator on $(X, \Sigma, \mu)$, which is normal on $L^2(X, \mu)$. Let $b$ be a slowly varying function, $\alpha \in [0, 1]$ and $B$ as above. Let $f \in L^2(X, \mu)$ satisfying $(C'_2)$. Then

(i) If $\alpha < 1/2$, \[ \frac{\sum_{k=1}^{n} T^k f}{n^{1-b(n)}} \xrightarrow{n \to +\infty} 0 \text{ $\mu$-a.s.} \]

(ii) If $\alpha = 1/2$, \[ \frac{\sum_{k=1}^{n} T^k f}{\sqrt{n} (\sum_{k=1}^{n} b(k))^{1/2}} \xrightarrow{n \to +\infty} 0 \text{ $\mu$-a.s.} \]

(iii) If $\alpha > 1/2$, \[ \frac{\sum_{k=1}^{n} T^k f}{\sqrt{n}} \xrightarrow{n \to +\infty} 0 \text{ $\mu$-a.s.} \]

**Remark 2.** As in Theorem 2.5, the rates obtained are optimal under condition $(C'_2)$. Our rate in (i) is essentially the same as that obtained by Gaposhkin [6] for unitary operators on $L^2(X, \mu)$.

In the case $p = 2$, it is also possible to give optimal conditions on $f$ to obtain a specific rate. For example, for $T$ induced by a measure-preserving transformation, we have

**Theorem 2.8.** Let $(X, \Sigma, \mu, \theta)$ be a dynamical system, with $\mu$ a probability. Let $b_0$ be any slowly varying function with $\sum_{n \geq 1} b_0(n)^2 < +\infty$. Then for every $f \in L^2(X, \mu)$ such that $\sum_{n \geq 1} \frac{\|S_n(f)\|^2}{n b_0(n)^2} < +\infty$, we have

$$\frac{1}{\sqrt{n}} S_n(f) \xrightarrow{n \to +\infty} 0 \text{ $\mu$-a.s.}$$

Moreover, the series $\sum_{n \geq 1} \frac{f \circ \theta^n}{n}$ converges $\mu$-a.s.

On the other hand, if $\theta$ is invertible and the system is ergodic and non-atomic, for every slowly varying function $b_1$ with $\sum_{n \geq 1} b_1(n)^2 = +\infty$, there exists a function $f \in L^2(X, \mu)$ such that $\sum_{n \geq 1} \frac{\|S_n(f)\|^2}{n b_1(n)^2} = +\infty$ (hence $\int_X f \, d\mu = 0$) and $\limsup \left| \frac{1}{\sqrt{n}} S_n(f) \right| = +\infty$ $\mu$-a.s.

By Proposition 2.4, with $\alpha = 1/2$, the convergence of $\sum_{n \geq 1} \frac{\|S_n(f)\|^2}{n b_0(n)^2}$ ($i \in \{0, 1\}$) is equivalent to the fact that $f \in (I - B)L^2(X, \mu)$ for the corresponding $b_1$. Theorem 2.8 then becomes a direct application of Theorem 2.5.

**Remark 3.** For example, take in Theorem 2.8, $b_0 = \frac{1}{\sqrt{\log(n) \log\log n}}$, for $e > 0$, and $b_1 = \frac{1}{\sqrt{\log(n) \log\log n}}$. Then the condition $\sum_{n \geq 2} \log n (\log \log n)^{1+e} \frac{\|S_n(f)\|^2}{n^2} < +\infty$ is sufficient for (3), but, in general, the condition

$$\sum_{n \geq 2} \log n \log \log n \frac{\|S_n(f)\|^2}{n^2} < +\infty$$

is not. Theorem 2.8 has applications in probability, see [3].

3. Proof of Theorem 2.5

Let $f \in (I - B)L^p(X, \mu)$. There exists $h \in L^p(X, \mu)$ such that $f = (I - B)h$, and we may and do assume that $h \in (I - T)L^p(X, \mu)$, since $T$ is mean ergodic on $L^p(X, \mu)$ and $B(1) = 1$. It therefore suffices to show that, for every $h \in (I - T)L^p(X, \mu)$

$$\frac{1}{n^{1/p}} \left( \sum_{k=1}^{n} \frac{b(k)^{p}}{k^{p}} \right)^{1/p} \sum_{k=1}^{n} T^{k}(I - B(T))h \xrightarrow{n \to +\infty} 0 \text{ $\mu$-a.s.}$$

(4)
For \( n \geq 1 \) write \( \sum_{k=1}^{n} T^k(I - B) = C_n - D_n - E_n \), where

\[
C_n = T + \sum_{m=2}^{n} \left( \sum_{k \geq m} \beta_k \right) T^m, \quad E_n = \sum_{k=1}^{n} \left( \sum_{m \geq 2n+1} \beta_{m-k} \right) T^m
\]

and

\[
D_n = \sum_{m=n+1}^{2n} \sum_{k=m-n}^{m-1} \beta_k T^m = \sum_{l=1}^{n} \left( \sum_{k=l}^{l+n-1} \beta_k \right) T^{l+n}.
\]

Hence it suffices to study separately the operator sequences \( \{C_n\} \), \( \{D_n\} \) and \( \{E_n\} \) on \( L^p(X, \mu) \). The first part of Theorem 2.5 will follow from the next propositions, which may be proved as in [5, Theorem 3.2].

**Proposition 3.1.** Let \( T \) be a Dunford–Schwartz operator on a probability space \( (X, \Sigma, \mu) \). Let \( \alpha \in ]0, 1[ \), \( b \) be any slowly varying function and \( B \) as above. Then, for every \( h \in L^p(X, \mu) \)

\[
\sup_{n \geq 1} \frac{|C_n(h)| + |E_n(h)|}{n^{1-\alpha} b(n)} < +\infty \quad \mu\text{-a.s.}
\]

**Proposition 3.2.** Let \( T \) be a Dunford–Schwartz operator on a probability space \( (X, \Sigma, \mu) \). Let \( \alpha \in ]0, 1[ \), \( b \) be any slowly varying function and \( B \) as above. Then, for every \( h \in L^p(X, \mu) \)

\[
\sup_{n \geq 1} \frac{|D_n(h)|}{n^{1/p} (\sum_{k=1}^{n} b(k)^{\alpha})^{1/p}} < +\infty \quad \mu\text{-a.s.}
\]

One can see that there exists \( K > 0 \) such that \( n^{1/p} (\sum_{k=1}^{n} b(k)^{\alpha})^{1/p} \geq Kn^{1-\alpha} b(n) \). Hence an application of Banach’s principle (see e.g. [7, Theorem 7.2a, p. 64]) yields that the set of functions of \( L^p(X, \mu) \) satisfying (4) is closed in \( L^p(X, \mu) \). It is not difficult to check that (4) is true for \( f \in (I - T)L^p(X, \mu) \), hence the first part of the theorem is proved.

Let us prove the second part of Theorem 2.5. By Banach’s principle (see [7, Theorem 7.2b, p. 64]), it suffices to show that there does not exist positive decreasing function \( \chi \) on \([0, +\infty[, \) with \( \lim_{\lambda \to +\infty} \chi(\lambda) = 0 \), such that for every \( f \in L^p(X, \mu) \) we have

\[
\mu \left( \left\{ x \in X: \sup_{n \geq 1} \left| \frac{\sum_{k=1}^{n} (I - B)f \circ \theta^k}{n^{1/p} \psi(n)} \right| \geq \lambda \| f \|_p \right\} \right) \leq \chi(\lambda) \quad \forall \lambda > 0.
\]

Hence it suffices to find \( \delta > 0 \), \( L_m \xrightarrow{m \to +\infty} +\infty \), and \( \{f_m\} \subset L^p(X, \mu) \) with \( \sup_{m \geq 1} \| f_m \|_p < +\infty \), such that

\[
\mu \left( \left\{ x \in X: \sup_{n \geq 1} \left| \frac{\sum_{k=1}^{n} (I - B)f_m \circ \theta^k}{n^{1/p} \psi(n)} \right| \geq L_m \right\} \right) \geq \delta \quad \forall m \geq 1.
\]

Using that \( b \) is slowly varying one can show that \( \beta_n \sim \frac{Cb(n)}{an^{1+\alpha}} \) and that there exists \( D > 0 \) such that for every \( n \geq 1 \) and \( l \in \{1, \ldots, n\} \), \( \sum_{k=l}^{l+n-1} \beta_k \geq D b(l) \). Hence for every non-negative measurable function \( f \) in \( L^p(X, \mu) \),

\[
D_n(f) \geq D \sum_{l=1}^{n} \frac{b(l)}{l^\alpha} f \circ \theta^{l+n}.
\]

The following construction is inspired by Déniel [4].

Let \( n \geq 1 \). By Rokhlin’s Lemma (see e.g. [8, Lemma 4.7, p. 48]), there exists a set \( Y_n \subset \Sigma \), such that the sets \( \{\theta^k(Y_n)\}_{1 \leq k \leq 2n} \) are disjoint and \( \mu(X - \bigcup_{k=1}^{2n} \theta^k(Y_n)) < \frac{1}{2n+1} \). In particular, for every \( k \in \{1, \ldots, 2n\} \), \( \frac{1}{2n+1} \leq \mu(Y_n) = \mu(\theta^k(Y_n)) \leq \frac{1}{2n} \).

For every \( n \geq 1 \), define \( u_n := \sum_{k=1}^{2n} b(k)^{\alpha} \) and a non-negative function \( f_n \) on \( X \) by \( f_n(x) = 0 \) for \( x \in X - \bigcup_{k=n+1}^{2n} \theta^k(Y_n) \), and \( f_n(x) = \left( \frac{b(k)^{\alpha}}{k^{\alpha}} \right)^{1/p} (\frac{1}{2n})^{1/p} \) if \( x \in \theta^k(Y_n) \), for some \( k \in \{n+1, \ldots, 2n\} \).
Then \( \{ f_n \} \) is bounded in \( L^p(X,\mu) \). Indeed, we have

\[
\| f_n \|_p = \sum_{k=n+1}^{2n} \| f_n \theta^k(Y_n) \|_p = \frac{n}{u_n} \sum_{k=n+1}^{2n} \left( \frac{b(k-n)}{(k-n)^\alpha} \right)^q \mu(Y_n) \leq 1/2.
\]

Let \( 0 \leq j \leq n - 1 \) and take \( x \in \theta^j(Y_n) \). Let \( y \in Y_n \), such that \( x = \theta^j(y) \). We have

\[
D_{n-j}(f)(x) \geq D \sum_{l=1}^{n-j} b(l) \theta^k(Y_n)(x) \geq D \sum_{l=1}^{n-j} b(l) \theta^{n+l}(y)
\]

\[
= D \left( \frac{n}{u_n} \right)^{1/p} \frac{n-j}{q/p} \left( \frac{b(l)}{l^{\alpha}} \right) = D \left( \frac{n}{u_n} \right)^{1/p} u_n^{-j}.
\]

Using that \( b \) is slowly varying, one can see that there exists \( K > 0 \) such that for every \( n \geq 1 \), and every \( j \leq n/2 \),

\[
u^{-j} \geq K u_n.
\]

Hence, noticing that \( C_{n-j}(f_n)(x) = 0 \) and \( E_{n-j}(f_n)(x) \geq 0 \), we obtain, for every \( 0 \leq j \leq n/2 \) and \( x \in \theta^j(Y_n) \),

\[
\left| \sum_{k=1}^{n-j} (I - B)(f_n)(\theta^k(x)) \right| \geq D \frac{u_n^{1/q}}{K^{1/p} \psi(n-j)} (n-j)^{1/p} \psi(n-j).
\]

So, on the set \( \bigcup_{0 \leq j \leq n/2} \theta^j(Y_n) \) whose measure is greater than \( \frac{n}{2n+1} \sim \frac{1}{3} \),

\[
\sup_{r \geq 1} \frac{\left| \sum_{k=1}^{r} (I - B)(f_n) \circ \theta^k \right|}{r^{1/p} \psi(r)} \geq \frac{D}{K^{1/p} \inf_{s \geq n/2} \psi(s)} u_n^{1/q} \rightarrow +\infty \text{ as } n \rightarrow +\infty,
\]

which proves (6).

References