

Mathematical Analysis/Functional Analysis

Lipschitz functions of perturbed operators

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Abstract

We prove that if f is a Lipschitz function on \mathbb{R} , and A and B are self-adjoint operators such that $\text{rank}(A - B) = 1$, then $f(A) - f(B)$ belongs to the weak space $S_{1,\infty}$, i.e., $s_j(A - B) \leq \text{const}(1 + j)^{-1}$. We deduce from this result that if $A - B$ belongs to the trace class S_1 and f is Lipschitz, then $f(A) - f(B) \in S_\Omega$, i.e., $\sum_{j=0}^n s_j(f(A) - f(B)) \leq \text{const} \log(2 + n)$. We also obtain more general results about the behavior of double operator integrals of the form $Q = \iint (f(x) - f(y))(x - y)^{-1} dE_1(x)T dE_2(y)$, where E_1 and E_2 are spectral measures. We show that if $T \in S_1$, then $Q \in S_\Omega$ and if $\text{rank} T = 1$, then $Q \in S_{1,\infty}$. Finally, if T belongs to the Matsaev ideal S_ω , then Q is a compact operator. **To cite this article:** F. Nazarov, V. Peller, C. R. Acad. Sci. Paris, Ser. I 347 (2009).

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Résumé

Fonctions lipschitziennes d'opérateurs perturbés. Nous démontrons que si f est une fonction lipschitzienne, A et B des opérateurs autoadjoints tels que $\text{rank}(A - B) = 1$, alors $f(A) - f(B) \in S_{1,\infty}$, c'est-à-dire $s_j(A - B) \leq \text{const}(1 + j)^{-1}$. Si $A - B$ est dans la classe S_1 des opérateurs à trace, nous montrons que $f(A) - f(B) \in S_\Omega$, c'est-à-dire $\sum_{j=0}^n s_j(f(A) - f(B)) \leq \text{const} \log(2 + n)$. Plus généralement, pour une fonction lipschitzienne f et pour des mesures spectrales E_1 et E_2 , considérons l'intégrale double opératorielle $Q = \iint (f(x) - f(y))(x - y)^{-1} dE_1(x)T dE_2(y)$. Nous montrons que si $T \in S_1$, alors $Q \in S_\Omega$ et si $\text{rank} T = 1$, alors $Q \in S_{1,\infty}$. Finalement, si T appartient à l'idéal de Matsaev S_ω , alors Q est un opérateur compact. **Pour citer cet article :** F. Nazarov, V. Peller, C. R. Acad. Sci. Paris, Ser. I 347 (2009).

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Version française abrégée

Dans cette Note nous considérons les propriétés de $f(A) - f(B)$, où f est une fonction lipschitzienne sur la droite réelle \mathbb{R} , A et B sont des opérateurs autoadjoints (pas nécessairement bornés) dont la différence $A - B$ est « petite ». Il est bien connu que si $A - B$ appartient à l'espace S_1 des opérateurs nucléaires, l'opérateur $f(A) - f(B)$ n'appartient pas nécessairement à S_1 .

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Nous démontrons que si $A - B \in \mathcal{S}_1$ et f est une fonction lipschitzienne, alors $f(A) - f(B)$ appartient à l'idéal \mathcal{S}_Ω défini comme l'ensemble d'opérateurs T dont les nombres singuliers $s_j(T)$ satisfont à l'inégalité

$$\sum_{j=0}^n s_j(T) \leq \text{const} \log(2+n), \quad n \geq 0.$$

Pour démontrer ce résultat nous utilisons la formule de Birman et Solomyak

$$f(A) - f(B) = \iint \frac{f(x) - f(y)}{x - y} dE_A(x)(A - B) dE_B(y),$$

où E_A et E_B sont les mesures spectrales des opérateurs A et B (la théorie des intégrales doubles opératorielles est développée dans les travaux [2,3] et [4] de Birman et Solomyak). Nous établissons un résultat plus général : si f est une fonction lipschitzienne, E_1 et E_2 des mesures spectrales et T un opérateur de la classe \mathcal{S}_1 , alors

$$\iint \frac{f(x) - f(y)}{x - y} dE_1(x)T dE_2(y) \in \mathcal{S}_\Omega.$$

Nous pouvons améliorer les résultats ci-dessus dans le cas $\text{rank } T = 1$. En réalité, dans ces cas

$$\iint \frac{f(x) - f(y)}{x - y} dE_1(x)T dE_2(y) \in \mathcal{S}_{1,\infty} \stackrel{\text{def}}{=} \left\{ T : \|T\|_{\mathcal{S}_{1,\infty}} \stackrel{\text{def}}{=} \sup_{j \geq 0} s_j(T)(1+j) < \infty \right\}.$$

Ce fait implique que si A et B sont des opérateurs autoadjoints tels que $\text{rank}(A - B) = 1$, alors $f(A) - f(B) \in \mathcal{S}_{1,\infty}$.

En utilisant des arguments de dualité on peut montrer que si T appartient à l'idéal de Matsaev \mathcal{S}_ω , c'est-à-dire

$$\sum_{j \geq 0} \frac{s_j(T)}{1+j} < \infty,$$

alors $\iint (f(x) - f(y))(x - y)^{-1} dE_1(x)T dE_2(y)$ est un opérateur compact. En particulier, si A et B sont des opérateurs autoadjoints tels que $A - B \in \mathcal{S}_\omega$, alors $f(A) - f(B)$ est un opérateur compact.

Pour établir les résultats ci-dessus nous montrons que si μ et ν sont des mesures boréliennes finies sur \mathbb{R} , $\varphi \in L^2(\mu)$, $\psi \in L^2(\nu)$,

$$k(x, y) = \varphi(x) \frac{f(x) - f(y)}{x - y} \psi(y), \quad x, y \in \mathbb{R},$$

et si $\mathcal{I}_k : L^2(\nu) \rightarrow L^2(\mu)$ est l'opérateur intégral défini par $(\mathcal{I}_k g)(x) = \int k(x, y)g(y) d\nu(y)$, alors

$$\sup_{j \geq 0} (1+j)s_j(\mathcal{I}_k) \leq \text{const} \|f\|_{\text{Lip}} \|f\|_{L^2(\mu)} \|\psi\|_{L^2(\nu)}.$$

En utilisant des arguments d'interpolation on peut démontrer que si T appartient à la classe de Schatten–von Neumann \mathcal{S}_p , $1 \leq p < \infty$, et $\varepsilon > 0$, alors

$$\iint (f(x) - f(y))(x - y)^{-1} dE_1(x)T dE_2(y) \in \mathcal{S}_{p+\varepsilon}.$$

En particulier, si A et B sont des opérateurs autoadjoints tels que $A - B \in \mathcal{S}_p$, alors $f(A) - f(B) \in \mathcal{S}_{p+\varepsilon}$.

La question de savoir si la condition $T \in \mathcal{S}_1$ implique que

$$\iint (f(x) - f(y))(x - y)^{-1} dE_1(x)T dE_2(y) \in \mathcal{S}_{1,\infty}$$

est toujours ouverte. Une réponse positive impliquerait que, dans le cas $1 < p < \infty$, on a $f(A) - f(B) \in \mathcal{S}_p$ pour toute paire d'opérateurs autoadjoints A, B dont la différence $A - B$ appartient à \mathcal{S}_p .

Finalement nous voudrions signaler qu'on peut obtenir des résultats similaires pour les fonctions d'opérateurs unitaires et pour les fonctions de contractions.

1. Introduction

In this Note we study the behavior of Lipschitz functions of perturbed operators. It is well known that if $f \in \text{Lip}$, i.e., f is a Lipschitz function and A and B are self-adjoint operators with difference in the trace class \mathcal{S}_1 , then $f(A) - f(B)$ does not have to belong to \mathcal{S}_1 . The first example of such f , A , and B was constructed in [5]. Later in [7] a necessary condition on f was found (f must be locally in the Besov space B_1^1) under which the condition $A - B \in \mathcal{S}_1$ implies that $f(A) - f(B) \in \mathcal{S}_1$. That necessary condition also implies that the condition $f \in \text{Lip}$ is not sufficient.

On the other hand, Birman and Solomyak showed in [4] that if $A - B$ belongs to the Hilbert–Schmidt class \mathcal{S}_2 , then $f(A) - f(B) \in \mathcal{S}_2$ and $\|f(A) - f(B)\|_{\mathcal{S}_2} \leq \|f\|_{\text{Lip}} \|A - B\|_{\mathcal{S}_2}$, where $\|f\|_{\text{Lip}} \stackrel{\text{def}}{=} \sup_{x \neq y} |f(x) - f(y)| \cdot |x - y|^{-1}$. Moreover, it was shown in [4] that in this case $f(A) - f(B)$ can be expressed in terms of the following double operator integral

$$f(A) - f(B) = \iint \frac{f(x) - f(y)}{x - y} dE_A(x)(A - B) dE_B(y), \tag{1}$$

where E_A and E_B are the spectral measures of A and B . We refer the reader to [2,3], and [4] for the beautiful theory of double operator integrals. Note that the divided difference $(f(x) - f(y))/(x - y)$ is not defined on the diagonal. Throughout this note we assume that it is zero on the diagonal.

In this note we study properties of the operators $f(A) - f(B)$ for (not necessarily bounded) self-adjoint operators A and B such that $A - B$ has rank one or $A - B \in \mathcal{S}_1$. Actually, we consider more general operators of the form

$$\mathcal{I}_{E_1, E_2}(f, T) \stackrel{\text{def}}{=} \iint \frac{f(x) - f(y)}{x - y} dE_1(x)T dE_2(y), \tag{2}$$

where E_1 and E_2 are Borel spectral measures on \mathbb{R} and $\text{rank } T = 1$ or $T \in \mathcal{S}_1$. Duality arguments also allow us to study double operator integrals (2) in the case when T belongs to the *Matsaev ideal* \mathcal{S}_ω .

Recall the definitions of the following operator ideals:

$$\mathcal{S}_{1, \infty} \stackrel{\text{def}}{=} \left\{ T : \|T\|_{\mathcal{S}_{1, \infty}} \stackrel{\text{def}}{=} \sup_{j \geq 0} s_j(T)(1 + j) < \infty \right\},$$

$$\mathcal{S}_\Omega \stackrel{\text{def}}{=} \left\{ T : \|T\|_{\mathcal{S}_\Omega} \stackrel{\text{def}}{=} \sup_n (\log(2 + n))^{-1} \sum_{j=0}^n s_j(T) < \infty \right\},$$

and

$$\mathcal{S}_\omega \stackrel{\text{def}}{=} \left\{ T : \|T\|_{\mathcal{S}_\omega} \stackrel{\text{def}}{=} \sum_{j=0}^{\infty} \frac{s_j(T)}{1 + j} < \infty \right\}.$$

It is well known that $\mathcal{S}_{1, \infty}$ is not a Banach space and its Banach envelope is contained in \mathcal{S}_Ω . Also recall that the dual space to \mathcal{S}_ω can be identified in a natural way with \mathcal{S}_Ω .

Note that the recent paper [1] contains results on properties of $f(A) - f(B)$ for f in the Hölder class A_α , $0 < \alpha < 1$, and self-adjoint operators A and B with $A - B$ in Schatten–von Neumann classes \mathcal{S}_p .

2. Main results

Theorem 2.1. *Let $f \in \text{Lip}$ and let E_1 and E_2 be Borel spectral measures on \mathbb{R} . If $\text{rank } T = 1$, then $\mathcal{I}_{E_1, E_2}(f, T) \in \mathcal{S}_{1, \infty}$ and*

$$\|\mathcal{I}_{E_1, E_2}(f, T)\|_{\mathcal{S}_{1, \infty}} \leq \text{const} \|f\|_{\text{Lip}} \|T\|.$$

Theorem 2.1 immediately implies the following result:

Theorem 2.2. *Let $f \in \text{Lip}$ and let E_1 and E_2 be Borel spectral measures on \mathbb{R} . If $T \in \mathcal{S}_1$, then $\mathcal{I}_{E_1, E_2}(f, T) \in \mathcal{S}_\Omega$ and*

$$\|\mathcal{I}_{E_1, E_2}(f, T)\|_{\mathcal{S}_\Omega} \leq \text{const} \|f\|_{\text{Lip}} \|T\|_{\mathcal{S}_1}.$$

By duality, we obtain the following theorem:

Theorem 2.3. *Let $f \in \text{Lip}$, and let E_1 and E_2 be Borel spectral measures on \mathbb{R} . Then the transformer $T \mapsto \mathcal{I}_{E_1, E_2}(f, T)$ defined on \mathcal{S}_2 extends to a bounded linear operator from \mathcal{S}_ω to the ideal of all compact operator and*

$$\|\mathcal{I}_{E_1, E_2}(f, T)\| \leq \text{const} \|f\|_{\text{Lip}} \|T\|_{\mathcal{S}_\omega}.$$

Using interpolation arguments, we can easily obtain from Theorem 2.2 the following fact:

Theorem 2.4. *Let $f \in \text{Lip}$, and let E_1 and E_2 be Borel spectral measures on \mathbb{R} . Suppose that $1 \leq p < \infty$ and $\varepsilon > 0$. If $T \in \mathcal{S}_p$, then $\mathcal{I}_{E_1, E_2}(f, T) \in \mathcal{S}_{p+\varepsilon}$.*

Birman–Solomyak formula (1) allows us to deduce straightforwardly from Theorems 2.1, 2.2, and 2.3 the following theorem:

Theorem 2.5. *Let A and B be self-adjoint operators on Hilbert space and let $f \in \text{Lip}$. We have*

- (i) *if $\text{rank}(A - B) = 1$, then $f(A) - f(B) \in \mathcal{S}_{1, \infty}$ and $\|f(A) - f(B)\|_{\mathcal{S}_{1, \infty}} \leq \text{const} \|f\|_{\text{Lip}} \|A - B\|$;*
- (ii) *if $A - B \in \mathcal{S}_1$, then $f(A) - f(B) \in \mathcal{S}_\Omega$ and $\|f(A) - f(B)\|_{\mathcal{S}_\Omega} \leq \text{const} \|f\|_{\text{Lip}} \|A - B\|_{\mathcal{S}_1}$;*
- (iii) *if $A - B \in \mathcal{S}_\omega$, then $f(A) - f(B)$ is compact and $\|f(A) - f(B)\| \leq \text{const} \|f\|_{\text{Lip}} \|A - B\|_{\mathcal{S}_\omega}$;*
- (iv) *if $1 \leq p < \infty$, $\varepsilon > 0$, and $A - B \in \mathcal{S}_p$, then $f(A) - f(B) \in \mathcal{S}_{p+\varepsilon}$.*

It is still unknown whether the assumption $T \in \mathcal{S}_1$ implies that $\mathcal{I}_{E_1, E_2}(f, T) \in \mathcal{S}_{1, \infty}$. If this is true, then the condition $A - B \in \mathcal{S}_p$ would imply that $f(A) - f(B) \in \mathcal{S}_p$ for $1 < p < \infty$.

To prove Theorem 2.1, we obtain a weak type estimate for Schur multipliers.

For a kernel function $k \in L^2(\mu \times \nu)$, we define the integral operator $\mathcal{I}_k : L^2(\nu) \rightarrow L^2(\mu)$ by

$$(\mathcal{I}_k g)(x) = \int k(x, y)g(y) \, d\nu(y), \quad g \in L^2(\nu).$$

As in the case of transformers from \mathcal{S}_1 to \mathcal{S}_1 (see [4]), Theorem 2.1 reduces to the following fact:

Theorem 2.6. *Let μ and ν be finite Borel measures on \mathbb{R} , $\varphi \in L^2(\mu)$, $\psi \in L^2(\nu)$. Suppose that $f \in \text{Lip}$ and the kernel function k is defined by*

$$k(x, y) = \varphi(x) \frac{f(x) - f(y)}{x - y} \psi(y), \quad x, y \in \mathbb{R}.$$

Then the integral operator $\mathcal{I}_k : L^2(\nu) \rightarrow L^2(\mu)$ with kernel function k belongs to $\mathcal{S}_{1, \infty}$ and

$$\|\mathcal{I}_k\|_{\mathcal{S}_{1, \infty}} \leq \text{const} \|f\|_{\text{Lip}} \|\varphi\|_{L^2(\mu)} \|\psi\|_{L^2(\nu)}.$$

Proof. Without loss of generality we may assume that $\|\varphi\|_{L^2(\mu)} = \|\psi\|_{L^2(\nu)} = 1$ and $\|f\|_{\text{Lip}} = 1$. Let us fix a positive integer n .

Given $N > 0$, we denote by P_N multiplication by the characteristic function of $[-N, N]$ (we use the same notation for multiplication on $L^2(\mu)$ and on $L^2(\nu)$). Then for sufficiently large values of N ,

$$\|\mathcal{I}_k - P_N \mathcal{I}_k P_N\|_{\mathcal{S}_2} < \frac{1}{n^{1/2}}. \tag{3}$$

Clearly, $P_N \mathcal{I}_k P_N$ is the integral operator with kernel function k_N , $k_N(x, y) = \chi_N(x)k(x, y)\chi_N(y)$, where $\chi_N = \chi_{[-N, N]}$ is the characteristic function of $[-N, N]$. We fix $N > 0$, for which (3) holds.

Consider now the points x_j , $1 \leq j \leq r$, and y_j , $1 \leq j \leq s$, at which μ and ν have point masses and

$$|\varphi(x_j)|^2 \mu\{x_j\} \geq \frac{1}{n}, \quad 1 \leq j \leq r, \quad \text{and} \quad |\psi(y_j)|^2 \nu\{y_j\} \geq \frac{1}{n}, \quad 1 \leq j \leq s. \tag{4}$$

Clearly, $r \leq n$ and $s \leq n$. We define now the kernel function k_{\sharp} by

$$k_{\sharp}(x, y) = u(x)k_N(x, y)v(y), \quad x, y \in \mathbb{R},$$

where

$$u(x) \stackrel{\text{def}}{=} 1 - \chi_{\{x_1, \dots, x_r\}}(x) \quad \text{and} \quad v(y) \stackrel{\text{def}}{=} 1 - \chi_{\{y_1, \dots, y_s\}}(y).$$

Obviously, the integral operators \mathcal{I}_{k_N} and $\mathcal{I}_{k_{\sharp}}$ coincide on a subspace of codimension at most $r + s \leq 2n$.

We can split now the interval $[-N, N]$ into no more than n subintervals I , $I \in \mathfrak{J}$, such that

$$\int_I |\varphi(x)|^2 u(x) \, d\mu(x) + \int_I |\psi(y)|^2 v(y) \, d\nu(y) \leq \frac{4}{n}, \quad I \in \mathfrak{J}.$$

This is certainly possible because of (4).

We have $\mathcal{I}_{k_{\sharp}} = \mathcal{I}^{(1)} + \mathcal{I}^{(2)} + \mathcal{I}^{(3)}$, where

$$(\mathcal{I}^{(1)}g)(x) = \int_{\mathbb{R}} \left(\sum_{I \in \mathfrak{J}} \chi_I(x)k_{\sharp}(x, y)\chi_I(y) \right) g(y) \, d\nu(y),$$

$$(\mathcal{I}^{(2)}g)(x) = \int_{\mathbb{R}} \left(\sum_{I, J \in \mathfrak{J}, I \neq J, |I| \geq |J|} \chi_I(x)k_{\sharp}(x, y)\chi_I(y) \right) g(y) \, d\nu(y),$$

and

$$(\mathcal{I}^{(3)}g)(x) = \int_{\mathbb{R}} \left(\sum_{I, J \in \mathfrak{J}, |I| < |J|} \chi_I(x)k_{\sharp}(x, y)\chi_I(y) \right) g(y) \, d\nu(y)$$

(we denote by $|I|$ the length of I). It is easy to see that $\|\mathcal{I}^{(1)}\|_{\mathcal{S}_2} \leq 4n^{-1/2}$. Let us estimate $\mathcal{I}^{(2)}$. The integral operator $\mathcal{I}^{(3)}$ can be estimated in the same way.

Suppose that $I, J \in \mathfrak{J}$, $I \neq J$, and $|I| \geq |J|$. For $x \in I$ and $y \in J$, we have

$$\frac{1}{x - y} = \frac{1}{x - c(J)} + \frac{y - c(J)}{x - c(J)} \cdot \frac{1}{x - y},$$

where $c(J)$ denotes the center of J .

Suppose that $g \perp \bar{\psi} \chi_J$ and $g \perp \bar{\psi} \bar{f} \chi_J$. Then $\mathcal{I}_2 g = \mathcal{I}_{k_b} g$, where

$$k_b(x, y) = \sum_{I, J \in \mathfrak{J}, I \neq J, |I| \geq |J|} u(x)\varphi(x)a_{IJ}(x, y)\psi(y)v(y)$$

and

$$a_{IJ}(x, y) = \chi_I(x) \frac{y - c(J)}{x - c(J)} \cdot \frac{f(x) - f(y)}{x - y} \chi_J(y).$$

Thus $\mathcal{I}^{(2)}$ and \mathcal{I}_{k_b} coincide on a subspace of codimension at most $2n$.

To estimate the Hilbert–Schmidt norm of \mathcal{I}_{k_b} , we observe that

$$|a_{IJ}(x, y)| \leq \frac{|J|}{(|J| + \text{dist}(I, J))}, \quad x \in I, \quad y \in J.$$

Thus

$$\begin{aligned} \|\mathcal{I}_{k_b}\|_{\mathcal{S}_2}^2 &\leq \sum_{I, J \in \mathfrak{J}, I \neq J, |I| \geq |J|} \left(\int_I |\varphi|^2 u \, d\mu \right) \left(\int_J |\psi|^2 v \, d\nu \right) \|a_{IJ}\|_{L^\infty}^2 \\ &\leq \frac{4}{n^2} \sum_{I, J \in \mathfrak{J}, I \neq J, |I| \geq |J|} \frac{|J|^2}{(|J| + \text{dist}(I, J))^2}. \end{aligned}$$

Let us observe that for a fixed $J \in \mathfrak{J}$,

$$\sum_{I \in \mathfrak{J}, I \neq J, |I| \geq |J|} \frac{|J|^2}{(|J| + \text{dist}(I, J))^2} \leq \text{const.} \quad (5)$$

Indeed, we can enumerate the intervals $I \in \mathfrak{J}$ satisfying $I \neq J$ and $|I| \geq |J|$ so that the resulting intervals I_k satisfy $\text{dist}(I_k, J) \leq \text{dist}(I_{k+1}, J)$. Since the intervals I_k are disjoint, we have

$$\text{dist}(I_k, J) \geq \frac{k-3}{2}|J|.$$

This easily implies (5). It follows that

$$\|\mathcal{I}_k\|_{\mathcal{S}_2}^2 \leq C \frac{4}{n^2} \cdot n = \frac{4C}{n}.$$

Similarly, $\mathcal{I}^{(3)}$ coincides on a subspace of codimension at most $2n$ with an operator whose Hilbert–Schmidt norm is at most $2(C/n)^{1/2}$.

If we summarize the above, we see that \mathcal{I}_k coincides on a subspace of codimension at most $6n$ with an operator whose Hilbert–Schmidt norm is at most $Kn^{-1/2}$, where K is a constant. Hence, on a subspace of codimension at most $7n$ the operator \mathcal{I}_k coincides with an operator whose norm is at most K/n , i.e.,

$$s_{7n}(\mathcal{I}_k) \leq \frac{K}{n}, \quad n \geq 1. \quad \square$$

Note that in the case of operators on the space $L^2(\mathbb{T})$ with respect to Lebesgue measure on the unit circle \mathbb{T} , the following related fact was obtained in [6] (see also [8]): if the derivative of f belongs to the Hardy class H^1 , φ and ψ belong to $L^\infty(\mathbb{T})$, and the kernel function k is defined by

$$k(\zeta, \tau) = \varphi(\zeta) \frac{f(\zeta) - f(\tau)}{\zeta - \tau} \psi(\tau), \quad \zeta, \tau \in \mathbb{T},$$

then the integral operator \mathcal{I}_k on $L^2(\mathbb{T})$ belongs to $\mathcal{S}_{1,2}$, i.e., $\sum_{j \geq 0} (s_j(\mathcal{I}_k))^2 (1+j) < \infty$.

To conclude the article, we note that similar results can be obtained for functions of unitary operators and for functions of contractions.

Remark. After this article had been written we have been informed by D. Potapov and F. Sukochev that they had proved the following result: if f is a Lipschitz function, $1 < p < \infty$, and A and B are self-adjoint operators such that $A - B \in \mathcal{S}_p$, then $f(A) - f(B) \in \mathcal{S}_p$.

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