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COMPTES RENDUS MATHEMATIQUE

Partial Differential Equations

Self-similar solutions with fat tails for a coagulation equation with nonlocal drift

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Abstract

We investigate the existence of self-similar solutions for a coagulation equation with nonlocal drift. In addition to explicitly given exponentially decaying solutions we establish the existence of self-similar profiles with algebraic decay. ***To cite this article: M. Herrmann et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Solutions auto-similaires à décroissance lente pour une équation de coagulation avec transport non local. L'existence de solutions auto-similaires préservant le volume total est établie pour une équation de coagulation incluant un terme de transport non local. Bien que cette équation admette des profils auto-similaires décroissant exponentiellement à l'infini, des profils auto-similaires avec une décroissance algébrique à l'infini sont construits lorsque le volume total est suffisamment petit. ***Pour citer cet article : M. Herrmann et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Version française abrégée

La théorie classique de Lifshitz et Slyozov [5] et Wagner [8] prédit l'évolution par fusion de la distribution des volumes d'une émulsion de gouttelettes et repose sur la description des interactions entre gouttelettes par un champ moyen commun $u = u(t)$. Plus précisément, la fonction de distribution $f = f(t, x) \geqslant 0$ des gouttelettes de volume $x > 0$ à l'instant $t \geqslant 0$ obéit à l'équation de transport non locale (1), le champ moyen u étant déterminé par la conservation du volume total. Les fonctions a et b décrivent les mécanismes d'échange de matière entre les gouttelettes et un choix possible est $a(x) = x^\alpha$ et $b(x) = x^\beta$ avec $0 \leqslant \beta < \alpha \leqslant 1$ (par exemple, $(\alpha, \beta) = (1/3, 0)$ [5]). Suite à l'analyse effectuée dans [5,8], le comportement en temps grands des solutions de (1) semble être de nature auto-similaire mais des articles récents ont montré que celui-ci dépend fortement du comportement local de la donnée initiale à l'extrémité

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supérieure de son support [1,2,7]. Pour remédier à cette « instabilité », une régularisation a été proposée dans [5] et consiste à prendre en compte la coalescence des gouttelettes, ce qui conduit à l'équation (2). Lorsque $a(x) = x^{1/3}$, $b(x) = 1$ et $w(x, y) = x + y$, l'existence d'une solution auto-similaire avec un profil à décroissance exponentielle a été récemment établie dans [3] mais l'obtention d'une classification complète des solutions auto-similaires semble difficile, ainsi que l'étude de leur stabilité éventuelle. En fait, ces questions sont pour l'instant non résolues pour l'équation de coagulation (correspondant au choix $a = b = 0$ dans (2)), sauf dans les deux cas suivants : $w(x, y) = 2$ et $w(x, y) = x + y$ [6]. Ces difficultés nous ont conduit à considérer le modèle simplifié (3) correspondant au choix $a(x) = x$, $b(x) = 1$ et $w(x, y) = 2$. Dans ce cas, les solutions auto-similaires à volume total constant sont de la forme $f(t, x) = t^{-2}F(x/t)$ et $u(t) = v/t$, où $v \in (0, \infty)$. La fonction F est alors une solution de l'équation (4) en la nouvelle variable $z = x/t$ avec $m_0 = \int_0^\infty F(z)\,\mathrm{d}z$, et $v = m_0/m_1$ est déterminé par la contrainte (5), le volume total $m_1 > 0$ étant fixé au préalable.

Il est facile de voir que, si $v \in (0, 1)$, $F_v(z) := v(1 - v)e^{-vz}$ est une solution de (4) avec $m_0 = 1 - v$ et $m_1 = (1 - v)/v$ et F_v décroit exponentiellement à l'infini. L'objet de cette Note est de montrer que F_v n'est pas le seul profil auto-similaire qui soit une solution de (4)–(5) et qu'il existe d'autres solutions de (4)–(5) dont la décroissance à l'infini est plus lente.

Théorème 1. *Soit $v \in (0, 1)$. Il existe $\overline{m_0} \in (0, 1 - v)$ telle que, pour chaque $m_0 \in (0, \overline{m_0}]$, le système* (4)–(5) *a une unique solution F avec les propriétés suivantes : $\int_0^\infty F(z)\,\mathrm{d}z = m_0$ et $m_1 = m_0/v$. De plus, la fonction F est positive, décroissante et $F(z) \sim cz^{-(2-v)/(1-v)}$ lorsque $z \to \infty$ pour une certaine constante $c > 0$.*

Il semble en fait raisonnable de conjecturer que, si $v \in (0, 1)$ est fixé, le système (4)–(5) a une unique solution pour chaque $m_0 \in (0, 1 - v]$ et aucune solution lorsque $m_0 > 1 - v$, mais nous ne sommes pas en mesure de le démontrer pour l'instant.

L'idée principale de la démonstration du Théorème 1 est de considérer la fonction $\tau(z) := -zF'(z)/F(z)$ qui est bien définie tant que $F(z) > 0$. Lorsque F est une solution de (4), la fonction τ est une solution de l'équation intégrale $\tau = H[F, \tau]$, l'opérateur H étant défini par (10). Lorsque m_0 est suffisamment petit, un argument de point fixe sur l'équation vérifiée par τ permet de construire une telle fonction ayant les propriétés requises et on en déduit l'existence de F.

1. Introduction

The classical mean-field theory by Lifshitz and Slyozov [5] and Wagner [8] describes the coarsening of droplets in dilute binary mixtures and is based on the assumption that droplets interact only via a common mean-field $u = u(t)$. It results in a nonlocal transport equation for the number density $f = f(t, x) \geqslant 0$ of droplets of volume $x > 0$ at time $t \geqslant 0$ and reads

$$\partial_t f(t,x) + \partial_x\big(\big(a(x)u(t) - b(x)\big)f(t,x)\big) = 0, \quad \int_0^\infty xf(t,x)\,\mathrm{d}x = m_1 = \text{const}, \tag{1}$$

where the second equation describes the conservation of matter (volume) and determines the mean-field u. The functions a and b are specified by the mechanism of transfer of matter between droplets; a typical example is $a(x) = x^\alpha$ and $b(x) = x^\beta$ with $0 \leqslant \beta < \alpha \leqslant 1$ (the original choice in [5] corresponding to $(\alpha, \beta) = (1/3, 0)$).

The large time behaviour of solutions to (1) was conjectured to be given by self-similar solutions already in [5,8] but it is now known that it actually depends sensitively on the details of the initial data at the end of their support [1,2,7]. As a regularisation it was suggested in [5] to add a coagulation term with additive kernel to the evolution equation for f which accounts for the occasional merging of droplets, that is,

$$\begin{aligned}\partial_t f(t,x) &+ \partial_x\big(\big(a(x)u(t) - b(x)\big)f(t,x)\big)\\ &= \frac{1}{2}\int_0^x w(x-y,y)f(t,x-y)f(t,y)\,\mathrm{d}y - f(t,x)\int_0^\infty w(x,y)f(t,y)\,\mathrm{d}y,\end{aligned} \tag{2}$$

the mean-field u still being given by the conservation of volume. Well-posedness for this case is proven in [4] and the existence of a fast decaying self-similar solution is established in [3] (for $a(x)=x^{1/3}$, $b(x)=1$, and $w(x,y)=x+y$); a full characterisation of all self-similar solutions seems however difficult, as well as the study of their stability. In fact, these questions are still open for the coagulation equation ($a=b=0$ in (2)), except for the so-called solvable kernels $w(x,y)=2$ and $w(x,y)=x+y$ [6]. In particular, besides the existence of exponentially decaying self-similar profiles, nothing is known in general about self-similar solutions with algebraic decay ("fat tails").

In order to develop methods to tackle these problems we consider here the following simplified version of (2)

$$\partial_t f(t,x)+\partial_x\big(\big(xu(t)-1\big)f(t,x)\big)=\int_0^x f(t,x-y)f(t,y)\,\mathrm{d}y-2f(t,x)\int_0^\infty f(t,y)\,\mathrm{d}y, \tag{3}$$

corresponding to the choice $a(x)=x$, $b(x)=1$, and $w(x,y)=2$ for $(x,y)\in(0,\infty)^2$. The function u is again specified by $\int_0^\infty xf(t,x)\,\mathrm{d}x=m_1$, which is equivalent to $u(t)=m_0/m_1$ with $m_0:=\int_0^\infty f(t,x)\,\mathrm{d}x$. Self-similar solutions to (3) are given by $f(t,x)=t^{-2}F(x/t)$ and $u(t)=v/t$, for some $v\in(0,\infty)$. Introducing $z=x/t$ we obtain that (F,v) solves

$$-\big(z(1-v)+1\big)F'(z)=(2-v-2m_0)F(z)+\int_0^z F(z-y)F(y)\,\mathrm{d}y,\quad z\in(0,\infty). \tag{4}$$

Here $m_0=\int_0^\infty F(z)\,\mathrm{d}z$, and v is such that for given $m_1>0$ the solution F satisfies

$$\int_0^\infty zF(z)\,\mathrm{d}z=m_1, \tag{5}$$

so (4) implies $v=m_0/m_1$. For the following analysis it is convenient to use m_0 and v as parameters. It is then easily seen that for each $v\in(0,1)$ there is an exponentially decaying solution

$$F_v(z)=m_0ve^{-vz}\quad\text{with}\quad m_0=1-v.$$

Besides these self-similar profiles which decay exponentially at infinity, we establish the existence of self-similar solutions with algebraic decay provided that the parameter m_0 is sufficiently small.

Theorem 1. *For each $v\in(0,1)$ there exists $\overline{m_0}\in(0,1-v)$ such that* (4)–(5) *has a unique solution F with $\int_0^\infty F(z)\,\mathrm{d}z=m_0$ and $m_1=m_0/v$ for all $m_0\in(0,\overline{m_0}]$. This solution F is nonnegative, non-increasing, and satisfies $F(z)\sim cz^{-(2-v)/(1-v)}$ as $z\to\infty$ for some $c>0$.*

In fact, we conjecture that for all $m_0\in(0,1-v)$ there is a unique solution to (4)–(5) as in Theorem 1 and that there is no solution with $m_0>1-v$. We aim to prove this conjecture in a future work by a continuation method starting from the solutions provided by Theorem 1.

2. Proof of the existence result

Within this section we always suppose that $v\in(0,1)$ is fixed. Furthermore we denote by $M_i(F):=\int_0^\infty y^iF(y)\,\mathrm{d}y$ with $i\in\mathbb{N}$ the ith moment of a nonnegative and integrable function F.

Basic properties of solutions. We start with deriving some elementary properties of solutions of (4)–(5). Integrating (4) we easily establish the following lemma:

Lemma 2. *Let F be a solution of* (4) *such that $M_0(F)=m_0$. Then we have $F(0)=m_0(1-m_0)$, and $m_1=M_1(F)<\infty$ implies $v=m_0/m_1$.*

This result in particular implies that once we have established the existence of a self-similar solution, then uniqueness follows by uniqueness of the corresponding initial-value problem. Moreover, we also infer from Lemma 2 that positive solutions can only exist for $m_0 \in (0,1)$, but for technical reasons, and since our results below requires m_0 to be sufficiently small anyway, we assume from now on that $m_0 < v/2$.

Since our existence proof relies on a fixed point argument for F we need appropriate supersolutions for (4). To this end we define $\alpha := (2 - v - 2m_0)/(1-v) \in (2,\infty)$ and the function $\overline{F}_{m_0}(z) := m_0\,(1+(1-v)z)^{-\alpha}$ for $z \geqslant 0$, which is the solution to the ordinary differential equation

$$-\big(z(1-v)+1\big)\overline{F}'_{m_0}(z) = (2-v-2m_0)\overline{F}_{m_0}(z), \quad \overline{F}_{m_0}(0) = m_0.$$

This function satisfies $\overline{F}_{m_0}(z) \sim cz^{-\alpha}$ as $z \to \infty$, and thanks to $m_0 < v/2$ we find $M_1(\overline{F}_{m_0}) < \infty$ as well as $\overline{F}'_{m_0}(z) \leqslant 0$ for all $z \geqslant 0$. As a consequence of the maximum principle for ordinary differential equations we readily derive the following comparison result:

Lemma 3. *Any nonnegative solution F to* (4) *with $F(0) \leqslant m_0$ satisfies $F(z) \leqslant \overline{F}_{m_0}(z)$ for all $z \geqslant 0$.*

From now on we restrict our considerations to *admissible* functions $F \in \mathcal{A}$, where $\mathcal{A}$ is the set of all nonnegative and continuous functions $F : [0,\infty) \to [0,\infty)$ with finite moments $M_0(F) < \infty$ and $M_1(F) < \infty$.

An auxiliary problem. A key ingredient in our existence proof is to show the existence of solutions to the following auxiliary problem: For a given $G \in \mathcal{A}$ with $M_0(G) = m_0$ we seek $F \in \mathcal{A}$ such that

$$-\big((1-v)z+1\big)F'(z) = (2-v-2m_0)F(z) + 2\int_0^{z/2} F(z-y)G(y)\,\mathrm{d}y, \quad M_0(F) = m_0. \tag{6}$$

Notice that the convolution operator in (6) is related to the integration over the interval $[0, z/2]$. For $F = G$, however, the identity $2\int_0^{z/2} F(z-y)F(y)\,\mathrm{d}y = \int_0^z F(z-y)F(y)\,\mathrm{d}y$ implies the equivalence of (6) and (4).

Lemma 4. *Consider $G \in \mathcal{A}$ with $M_0(G) = m_0$ and suppose that $F \in \mathcal{A}$ solves* (6)*. Then F is unique, monotonically decreasing, and fulfils $m_0 - 2m_0^2 \leqslant F(0) \leqslant m_0$.*

Proof. The uniqueness of solutions follows from the homogeneity of the problem combined with a Gronwall-like argument, and the monotonicity is implied by $F \geqslant 0$, $G \geqslant 0$, $v < 1$, and $2 - v - 2m_0 \geqslant 0$. To derive the bounds for the initial value we integrate over $z \in (0,\infty)$ to obtain

$$F(0) + (1-v)m_0 = (2-v)m_0 - 2m_0^2 + M_0(F*G) \quad \text{with } (F*G)(z) := 2\int_0^{z/2} F(z-y)G(y)\,\mathrm{d}y,$$

and the estimates

$$0 \leqslant M_0(F*G) = 2\int_0^{\infty}\int_0^{z/2} F(z-y)G(y)\,\mathrm{d}y\,\mathrm{d}z = 2\int_0^{\infty}\int_0^{z} G(y)F(z)\,\mathrm{d}y\,\mathrm{d}z \leqslant 2m_0^2$$

complete the proof. □

For the subsequent considerations it is convenient to introduce the function $\tau : [0,\infty) \to \mathbb{R}$ defined by

$$\tau(z) := -z\frac{F'(z)}{F(z)} \quad \text{as long as} \quad F(z) > 0, \tag{7}$$

so that

$$F(z) = F(0)\exp\left(-\int_0^z \frac{\tau(s)}{s}\,\mathrm{d}s\right). \tag{8}$$

Notice in particular that $\tau(z) \to \tau_\infty \neq 0$ as $z \to \infty$ implies $F(z) \sim cz^{-\tau_\infty}$ as $z \to \infty$. Rewriting (6) in terms of τ yields

$$\frac{(1-v)z+1}{z}\tau(z) = 2 - v - 2m_0 + h[G,\tau](z), \quad h[G,\tau](z) := 2\int_0^{z/2} G(y)\exp\left(\int_{z-y}^{z} \frac{\tau(s)}{s}\,\mathrm{d}s\right)\mathrm{d}y, \tag{9}$$

and this is equivalent to the fixed point equation

$$\tau = H[G,\tau], \quad H[G,\tau](z) := \frac{z}{(1-v)z+1}\big(2 - v - 2m_0 + h[G,\tau](z)\big). \tag{10}$$

In the following lemma we summarise some useful properties of the operator H.

Lemma 5. *For each $G \in \mathcal{A}$ the operator $\tau \mapsto H[G,\tau]$ is well defined on the set $C([0,\infty))$ and has the following properties*:

1. $0 \leqslant H[G,\tau]$ *for all* τ,
2. $\tau_1 \leqslant \tau_2$ *implies* $H[G,\tau_1] \leqslant H[G,\tau_2]$,
3. $\tau(z) \to \tau_\infty$ *as* $z \to \infty$ *implies* $H[G,\tau](z) \to (2-v)/(1-v)$ *as* $z \to \infty$.

Proof. The first two claims are direct consequences of $G \geqslant 0$ combined with $v < 1$ and $m_0 \leqslant v/2 < 1/2$, and the definitions (9) and (10). Now suppose that $\tau(z) \to \tau_\infty$, and write

$$h[G,\tau](z) = 2\int_0^{z/2} G(y)\exp\left(\int_{z-y}^{z} \frac{\tau(s)-\tau_\infty}{s}\,\mathrm{d}s\right)\left(\frac{z}{z-y}\right)^{\tau_\infty}\mathrm{d}y.$$

Due to $y \leqslant z/2$ the convergence assumption implies

$$\left|\exp\left(\int_{z-y}^{z} \frac{\tau(s)-\tau_\infty}{s}\,\mathrm{d}s\right) - 1\right| \leqslant \exp\Big(\ln 2 \sup_{s \geqslant z/2}\big|\tau(s)-\tau_\infty\big|\Big) - 1 \xrightarrow{z\to\infty} 0,$$

and from $M_0(G) = m_0 < \infty$, the inequality $z/(z-y) \leqslant 2$ for $y \in (0, z/2)$, and the Lebesgue Dominated Convergence Theorem we infer that

$$\lim_{z\to\infty} 2\int_0^{z/2} G(y)\left(\frac{z}{z-y}\right)^{\tau_\infty}\mathrm{d}y = 2\int_0^{\infty} G(y)\,\mathrm{d}y = 2m_0.$$

Hence, we have $h[G,\tau](z) \to 2m_0$ as $z \to \infty$, and the proof is complete. □

Solutions to the auxiliary problem. In order to set up an iteration scheme for τ we prove that for all sufficiently small m_0 there exists a supersolution to the fixed point equation (10). Notice that the constants in the next lemma depend on v.

Lemma 6. *There exist two constants $\overline{m_0} \in (0, v/2)$ and $\tau_\star \in \mathbb{R}$ such that $H[G,\tau_\star] \leqslant \tau_\star$ holds for all $G \in \mathcal{A}$ with $m_0 = M_0(G) \leqslant \overline{m_0}$.*

Proof. The definition (9) gives

$$h[G,\tau](z) \leqslant 2\int_0^{z/2} G(y)\left(\frac{z}{z-y}\right)^{\|\tau\|_\infty}\mathrm{d}y \leqslant 2m_0 2^{\|\tau\|_\infty}$$

and hence $H[G,\tau] \leqslant a_{m_0} + b_{m_0}2^{\|\tau\|_\infty}$ with $a_{m_0} := (2-v-2m_0)/(1-v)$ and $b_{m_0} := 2m_0/(1-v)$. Since $b_{m_0} \to 0$ as $m_0 \to 0$ we choose $\overline{m_0} > 0$ and $\sigma_\star > 0$ such that $b_{\overline{m_0}}2^{a_0} \leqslant \sigma_\star 2^{-\sigma_\star}$, and set $\tau_\star = a_0 + \sigma_\star$. Then, for all $m_0 \in (0, \overline{m_0}]$ we find $a_{m_0} + b_{m_0}2^{\tau_\star} \leqslant a_0 + b_{\overline{m_0}}2^{a_0}2^{\sigma_\star} \leqslant a_0 + \sigma^\star = \tau_\star$, and this implies the claimed result. □

Now we are able to prove the existence of solutions to the auxiliary problem:

Corollary 7. *For each $G \in \mathcal{A}$ with $m_0 = M_0(G) \in (0, \overline{m_0}]$ there exists a unique solution $F \in \mathcal{A}$ to* (6) *such that the function τ given by* (7) *satisfies $0 \leqslant \tau(z) \leqslant \tau_\star$ for all $z \geqslant 0$ and $\tau(z) \to (2 - v)/(1 - v)$ as $z \to \infty$. In addition, $F \leqslant \overline{F}_{m_0}$.*

Proof. We define a sequence of continuous functions $(\tau_i)_i$ by $\tau_0 = \tau_\star$ and $\tau_{i+1} = H[G, \tau_i]$, which satisfies $0 \leqslant \tau_{i+1} \leqslant \tau_i \leqslant \tau_\star$ thanks to Lemma 5 and Lemma 6. Consequently, this sequence converges pointwisely to a function τ, which solves the fixed point equation (10), and is hence continuous. The function F is then determined by (8) and the condition $M_0(F) = m_0$, the latter being meaningful as the behaviour of τ as $z \to \infty$ guarantees the integrability of F. Moreover, F satisfies the first equation in (6) by construction and $F \leqslant \overline{F}_{m_0}$ due to Lemma 4, and this implies $M_1(F) < \infty$. □

Fixed point argument for F. We finish the proof of Theorem 1 by applying Schauder's Fixed Point Theorem. For $m_0 \in (0, \overline{m_0}]$ let $\mathcal{M}_{m_0}$ denote the set of all functions $G \in \mathcal{A}$ that satisfy $M_0(G) = m_0$ and $G \leqslant \overline{F}_{m_0}$. In view of Lemma 4 and Corollary 7, we may define an operator $\mathcal{S}_{m_0} : \mathcal{M}_{m_0} \to \mathcal{M}_{m_0}$ as follows: For given $G \in \mathcal{M}_{m_0}$ the function $\mathcal{S}_{m_0}[G]$ is the solution to the auxiliary problem (6) with datum G.

Lemma 8. *For each $m_0 \in (0, \overline{m_0}]$ the operator $\mathcal{S}_{m_0}$ has a unique fixed point in $\mathcal{M}_{m_0}$, which satisfies both* (4) *and* (5) *with $m_1 = m_0/v$.*

Proof. From (6) and Lemma 4 we infer that each $F \in \mathcal{S}[\mathcal{M}_{m_0}]$ is differentiable with

$$\|F'\|_\infty \leqslant (2 - v + 2m_0)\|F\|_\infty \leqslant (2 - v + 2m_0)m_0.$$

Therefore, $\mathcal{S}_{m_0}[\mathcal{M}_{m_0}]$ is precompact in $C([0, \infty))$, and due to the uniform supersolution $\overline{F}_{m_0}$ we readily verify that the integral constraint $M_0(G) = m_0$ is preserved under strong convergence in $\mathcal{M}_{m_0}$. Schauder's theorem now implies the existence of a fixed point $F = \mathcal{S}_{m_0}[F] \in \mathcal{M}_{m_0}$, which satisfies (4) and $M_0(F) = m_0$ by construction. Moreover, $F \leqslant \overline{F}_{m_0}$ implies $M_1(F) < \infty$, so Lemma 2 guarantees (5). Finally, the fixed point is unique as discussed in the remark to Lemma 2. □

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