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A property of light-cones in Einstein's gravity

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Abstract

We prove that the area of cross-sections of future light-cones in a space–time of arbitrary dimension, solution of the Einstein equations with sources satisfying suitable energy conditions, is smaller than the area of corresponding cross-sections for Minkowskian cones, equal only in space–times which are Minkowskian to the future of the light-cone. ***To cite this article: Y. Choquet-Bruhat et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Une propriété des cones de lumière en gravitation Einsteinienne. On montre que l'aire des sections des cônes de lumière futurs dans un espace–temps de dimension quelconque, solution des équations d'Einstein avec sources obéissant à des conditions d''énergie appropriées, est plus petite que l'aire des sections correspondantes d'un cône Minkowskien, égale seulement dans les espace–temps Minkowskiens dans le futur du cône. ***Pour citer cet article : Y. Choquet-Bruhat et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

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Définitions

On considère les équations d'Einstein pour une métrique Lorentzienne g sur un espace–temps V de dimension $n+1$ avec second membre un tenseur d'impulsion énergie T

$$S := \mathrm{Ricci}(g) - \frac{R(g)}{2} g = T. \tag{1}$$

Elles s'écrivent en coordonnées locales, avec $\partial_\lambda := \frac{\partial}{\partial x^\lambda}$ et $\Gamma^\lambda_{\alpha\beta}$ les symboles de Christoffel,

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$$R_{\alpha\beta} \equiv \partial_\lambda \Gamma^\lambda_{\alpha\beta} - \partial_\alpha \Gamma^\lambda_{\beta\lambda} + \Gamma^\lambda_{\alpha\beta}\Gamma^\mu_{\lambda\mu} - \Gamma^\lambda_{\alpha\mu}\Gamma^\mu_{\beta\lambda} = \rho_{\alpha\beta}, \quad \rho_{\alpha\beta} := T_{\alpha\beta} - \frac{1}{n-1} g_{\alpha\beta} T^\lambda{}_\lambda. \tag{2}$$

On décompose Ricci(g) en opérateurs tensoriels en munissant V d'une métrique connue $\hat{g}$; on note $\hat{\Gamma}^\lambda_{\alpha\beta}$ les symboles de Christoffel et D la dérivée covariante associés à $\hat{g}$. On a

$$R_{\alpha\beta} \equiv R^{(H)}_{\alpha\beta} + \frac{1}{2}\big(g_{\alpha\lambda} D_\beta H^\lambda + g_{\beta\lambda} D_\alpha H^\lambda\big), \tag{3}$$

où H est le vecteur, appelé vecteur de jauge, de composantes

$$H^\lambda \equiv \Gamma^\lambda - W^\lambda, \quad \Gamma^\lambda := g^{\alpha\beta}\Gamma^\lambda_{\alpha\beta},\ W^\lambda := g^{\alpha\lambda}\hat{\Gamma}^\lambda_{\alpha\beta} \tag{4}$$

et $R^{(H)}_{\alpha\beta}$ est un opérateur quasi-linear, quasi-diagonal sur g, hyperbolique si g est Lorentzienne,

$$R^{(H)}_{\alpha\beta} \equiv -\frac{1}{2} g^{\lambda\mu} D_\lambda D_\mu g_{\alpha\beta} + Q_{\alpha\beta}, \tag{5}$$

Q est une forme quadratique des dérivées Dg, de coefficients analytiques en g et son associé contravariant.

Problème de Cauchy avec données sur un cône caractéristique C

Le problème est bien posé[1] pour les équations $R^{(H)}_{\alpha\beta} = \rho_{\alpha\beta}$. La donnée pour ces équations est la valeur $\bar{g}$ sur C des composantes du tenseur métrique. L'utilisation des identités de Bianchi montre que la solution, $g^{(H)}$, vérifie les équations (1) si $\bar{H} = 0$, i.e. si H s'annule sur C, ce que l'on démontre être vrai si $\bar{g}$ vérifie des contraintes [1]. Réciproquement, on peut démontrer que toute solution g de (1) est isométrique dans un voisinage futur du sommet de C à une solution $g^{(H)}$ avec $\bar{g}^{(H)}$ ne dépendant que de $\bar{g}$ [2].

Théorème du cône isotrope

On suppose que T satisfait la *condition d'énergie dominante*, $T(X,Y) \geqslant 0$ pour chaque paire de vecteurs causaux de même orientation temporelle.

En coordonnées adaptées les contraintes forment une hiérarchie d'équations différentielles ordinaires[2] dans la direction des géodésiques isotropes qui engendrent le cône C. On déduit de la première (équation de Raychauduri) que l'aire d'une section de C, à paramètre affine donné, est au plus égale à l'aire correspondante dans un cône Minkowskien. On utilise les suivantes pour démontrer que ces aires ne sont égales que dans le vide, pour l'espace–temps de Minkowski.

1. Adapted coordinates on a null cone

Let C_O be a hypersurface in $\mathbb{R}^{n+1}$, smooth except at the point O, which will be a null cone of vertex O in a space–time (V, g) with V some domain of $\mathbb{R}^{n+1}$. It is known that C_O is generated by geodesic null curves, called rays; the equation of C_O can be taken to be $x^0 = 0$ and we can choose coordinates x^1, x^A, $A = 2, \dots, n$, such that, on C_O, x^1 is a parameter along the rays and the subspaces $\Sigma_{x^1} := \{x^1 = \text{constant}\}$ are spacelike and diffeomorphic to the sphere S^{n-1}, except for Σ_0 which reduces to the point O. The space–time metric, $g := g_{\alpha\beta}\,dx^\alpha\,dx^\beta$, is assumed to be smooth, it reduces on C_O in such coordinates to[3]

$$\bar{g} := g|_{x^0=0} \equiv \bar{g}_{00}\big(dx^0\big)^2 + 2\nu_0\,dx^0\,dx^1 + 2\nu_A\,dx^0\,dx^A + \tilde{g}, \tag{6}$$

where $\nu_0 := \bar{g}_{01}$, $\nu_A := \bar{g}_{0A}$, $\tilde{g} := \bar{g}_{AB}\,dx^A\,dx^B$ are respectively an x^1-dependent scalar, 1-form, and Riemannian metric on S^{n-1}. We adorn with a tilde geometric elements pertaining to the metric $\tilde{g}$.

[1] Voir [4] et les références y incluses.

[2] Démonstration esquissée dans [7] puis développée explicitement dans [3] dans le cas $n = 3$ avec des données initiales portées par 2 surfaces caractéristiques sécantes et des coordonnées harmoniques.

[3] We put an overbar on the restriction to C_O of space–time quantities when it could be ambiguous.

We consider a Minkowski metric η for which C_O is a null cone and which reads

$$\eta := -\left(\mathrm{d}x^0\right)^2 + 2\,\mathrm{d}x^0\,\mathrm{d}x^1 + \left(x^1\right)^2 s_{AB}\,\mathrm{d}x^A\,\mathrm{d}x^B, \tag{7}$$

with $s_{AB}\,\mathrm{d}x^A\,\mathrm{d}x^B$ the metric of the round sphere S^{n-1}. It has the form (6) on C_O.

Coordinates are chosen such that $\bar{g}_{00}$, ν_0, ν_A and $(x^1)^{-2}g_{AB}$ tend to the Minkowskian values as x^1 tends to zero.

We take η as target $\hat{g}$ of our wave map. The non-zero Christoffel symbols of η are

$$\hat{\Gamma}^B_{1A} = \frac{1}{x^1}\delta^B_A, \quad \hat{\Gamma}^B_{AC} \equiv S^B_{AC}, \quad \hat{\Gamma}^0_{AB} = -x^1 s_{AB}, \quad \hat{\Gamma}^1_{AB} = -x^1 s_{AB}, \tag{8}$$

where the S^B_{AC}'s are the Christoffel symbols of s_{AB}.

Null extrinsic curvature

For an arbitrary choice of the parameter x^1, and of ν_0, we define an x^1 dependent tensor χ on Σ_{x^1}

$$\chi_{AB} := \overline{\nabla_A \ell_B} \equiv -\bar{\Gamma}^0_{AB}\nu_0 \equiv \frac{1}{2}\partial_1 g_{AB}, \quad \chi_A^{\ B} := \bar{g}^{BC}\chi_{AC}, \quad \tau := \chi_A^{\ A}. \tag{9}$$

χ is called the null extrinsic curvature of C_O if $\ell := \frac{\partial}{\partial x^1}$ is parallelly transported,[4] i.e.

$$\overline{\ell^\alpha \nabla_\alpha \ell^\beta} \equiv \overline{\nabla_1 \ell^\beta} \equiv \bar{\Gamma}^\beta_{11} \equiv \delta^\beta_1 \nu^0\left(\partial_1\nu_0 - \frac{1}{2}\overline{\partial_0 g_{11}}\right) = 0; \tag{10}$$

the coordinate x^1 is then an affine parameter on the null ray. The condition (10) depends on a derivative of the metric transversal to C_O. We impose that ν_0, satisfies[5]

$$\partial_1\nu_0 = \frac{1}{2}\nu_0(\bar{W}_1 + \tau), \quad \bar{W}_1 \equiv \nu_0\bar{W}^0. \tag{11}$$

We then have the lemma, which applies to metrics in wave gauge:

Lemma 1.1. *If $\bar{H}_1 \equiv \nu_0\bar{H}^0 = 0$, then ℓ satisfies* (10) *and x^1 is an affine parameter.*

2. Einstein-wave gauge constraints

The vector $\bar{S}_{\alpha\beta}\ell^\beta$ provides a generalization of the usual spacelike constraints operator to the case of a null hypersurface M with generator (and normal) ℓ. Since this operator contains derivatives transversal to M, which do not appear in the Cauchy data $\bar{g}$ on M, some further considerations are needed. We prove in [2] a basic theorem, generalization of previous ones in [7,3] for the case $n = 3$ and harmonic coordinates.

Theorem 2.1. *The operator $\bar{S}_{\alpha\beta}\ell^\beta$ on a null submanifold M is a sum $\mathcal{L}_\alpha + \mathcal{C}_\alpha$, where $\mathcal{L}_\alpha$ is an ordinary differential operator along the generators acting on the values on M of the wave gauge vector $\bar{H}$, vanishing for solutions in wave gauge. The operator $\mathcal{C}_\alpha$ depends only on the restrictions to M of the components of the space–time metric and on the pseudo vector $\bar{W}$ which depends on the target space of the gauge wave map. The operators $\mathcal{C}_\alpha$, called null Einstein-wave gauge constraints operators, appear in adapted null coordinates on the cone as a hierarchy of ordinary differential operators*[6] *along the generators, all linear when the first constraint, corresponding to $\bar{S}_{\alpha\beta}\ell^\alpha\ell^\beta$, has been solved.*

We will give in the following the explicit expressions of $\mathcal{C}_\alpha$ and $\mathcal{L}_\alpha$ in the case of interest to us, and the consequences for the solution of the constraints for metrics in wave gauge, in which

$$\mathcal{C}_\alpha = \bar{T}_{\alpha\beta}\ell^\beta.$$

[4] See for instance [5].
[5] Generalizing a choice made by Damour and Schmidt [3] for harmonic coordinates and $n = 3$.
[6] The final expressions have been obtained with the algebraic manipulations program *xAct* [6].

3. $\mathcal{C}_1$ constraint

Straightforward calculation gives the identity

$$\ell^\alpha \ell^\beta \bar{S}_{\alpha\beta} \equiv \bar{S}_{11} \equiv \bar{R}_{11} \equiv -\partial_1 \tau + \nu^0 \partial_1 \nu_0 \tau - \frac{1}{2}\tau(\bar{\Gamma}_1 + \tau) - \chi_A{}^B \chi_B{}^A. \tag{12}$$

The definition $\bar{H}_1 \equiv \bar{\Gamma}_1 - \bar{W}_1$ and the choice (11) show that

$$\bar{R}_{11} \equiv \mathcal{C}_1 + \mathcal{L}_1, \tag{13}$$

with

$$\mathcal{C}_1 \equiv -\partial_1 \tau - \chi_A{}^B \chi_B{}^A, \qquad \mathcal{L}_1 \equiv -\frac{1}{2}\tau \bar{H}_1. \tag{14}$$

We denote by σ the traceless part of χ. The $\mathcal{C}_1$ Einstein constraint reads

$$\mathcal{C}_1 \equiv -\partial_1 \tau - \frac{1}{n-1}\tau^2 - |\sigma|^2 = \bar{T}_{11}, \quad |\sigma|^2 := \sigma_A{}^B \sigma_B{}^A; \tag{15}$$

it is a first order differential equation for τ when σ and $\bar{T}_{11}$ are known. When $\sigma = \bar{T}_{11} = 0$ it admits as solution the mean null curvature of the Minkowskian null cone $\hat{\tau} = \frac{n-1}{x^1}$. In the general case we set $\tau - \hat{\tau} = \theta$. Eq. (15) becomes

$$\partial_1 \theta + \frac{2}{x^1}\theta = -|\sigma|^2 - \frac{\theta^2}{n-1} - \bar{T}_{11}. \tag{16}$$

It implies, since $\bar{T}_{11} \geqslant 0$ by the dominant energy assumption,

$$\partial_1 \{ (x^1)^2 \theta \} \leqslant 0.$$

Since $(x^1)^2\theta$ tends to zero as x^1 tends to zero, we conclude that $\theta \leqslant 0$, and $\theta = 0$ if and only if $|\sigma|^2 = 0$ and $\bar{T}_{11} = 0$.

We denote by $\mathcal{A}(s)$ the volume of the section $x^1 = s$ of the cone C_O,

$$\mathcal{A}(s) := \int_{S^{n-1}} \sqrt{\det \tilde{g}|_{x^1=s}}\, \mathrm{d}x^2 \cdots \mathrm{d}x^n,$$

then

$$\frac{\partial}{\partial x^1}\mathcal{A}(x^1) = \int_{S^{n-1}} \frac{\partial}{\partial x^1}\sqrt{\det \tilde{g}}\, \mathrm{d}x^2 \cdots \mathrm{d}x^n, \quad \text{with } \frac{\partial}{\partial x^1}\{\log \sqrt{\det \tilde{g}}\} = \tau.$$

By our definitions $s = x^1$ is an affine parameter such that

$$\lim_{x^1 \to 0} \frac{\sqrt{\det \tilde{g}}}{\sqrt{\det \tilde{g}_0}} = 1,$$

where $\tilde{g}_0$ is the corresponding metric in Minkowski space–time. Such parameters will be called *normalised*. Then, if $\theta \leqslant 0$ it holds that, with $\mathcal{A}_0(s)$ the volume of the section $x^1 = s$ of the null cone of the Minkowski metric (7),

$$\frac{\partial}{\partial x^1}\mathcal{A}(x^1) \leqslant \frac{\partial}{\partial x^1}\mathcal{A}_0(x^1), \quad \text{hence} \quad \mathcal{A}(x^1) \leqslant \mathcal{A}_0(x^1) = c_{n-1}(x^1)^{n-1}, \tag{17}$$

c_{n-1} the volume of the unit sphere S^{n-1}. The equality is satisfied if and only if

$$|\sigma|^2 \equiv 0 \quad \text{and} \quad \bar{T}_{11} \equiv 0, \quad \text{i.e.} \quad \bar{T}(\ell, \ell) \equiv 0. \tag{18}$$

We have just proved:

Theorem 3.1. *The area of a section Σ_s of a characteristic cone at normalised affine parameter distance s from the vertex in an Einsteinian space–time with source a stress energy tensor T which satisfies the dominant energy condition is smaller than or equal to the corresponding area of a Minkowskian cone.*

We will exploit the equalities (18) in what follows, starting with:

Lemma 3.2. *If $\sigma \equiv 0$ and $\tau = \frac{n-1}{x^1}$, then $\bar{g}_{AB} = (x^1)^2 s_{AB}$ and $\nu_0 = 1$.*

Proof. The equality $\bar{g}_{AB} = (x^1)^2 s_{AB}$ results from the definition of χ_{AB} and the assumption on the coordinates at O. Eq. (11) gives then

$$\partial_1 \nu_0 = \frac{1}{2}\nu_0(1-\nu_0)\frac{n-1}{x^1} \tag{19}$$

which implies $\nu_0 = 1$ if ν_0 tends to 1 when x^1 tends to zero. □

4. $\mathcal{C}_A$ constraint

Lemma 4.1. *If T satisfies the dominant energy condition then $\bar{T}_{11} = 0$ implies $\bar{T}_{1A} = 0$.*

Proof. $\bar{T}_{\alpha\beta}\ell^\alpha$ is causal by hypothesis on T, therefore $\bar{T}_{11} \equiv \bar{T}_{\alpha\beta}\ell^\alpha\ell^\beta = 0$ implies $\bar{T}_{\alpha\beta}\ell^\alpha = k\ell_\beta$, hence in our null coordinates $\bar{T}_{1A} = 0$. □

Straightforward computations, using the identities

$$\bar{W}_A := \bar{g}_{AB}\bar{W}^B \equiv -\frac{2}{x^1}\nu_A + S_A, \qquad S_A := s_{AB}\hat{g}^{CD}\hat{\Gamma}^B_{CD}, \tag{20}$$

$$\overline{\partial_0 g_{1A}} \equiv -\partial_1\nu_A - \nu_A\frac{n-1}{x^1} + \bar{H}_A, \tag{21}$$

lead to the identity $\bar{R}_{1A} \equiv \mathcal{C}_A + \mathcal{L}_A$, with

$$\mathcal{C}_A \equiv \partial_1\partial_1\nu_A + \frac{3n-5}{2}\frac{\partial_1\nu_A}{x^1} + \frac{(n-2)(n-3)}{2}\frac{\nu_A}{(x^1)^2},$$
$$\mathcal{L}_A \equiv -\frac{1}{2}\left(\partial_1\bar{H}_A + \frac{n-1}{x^1}\bar{H}_A\right). \tag{22}$$

For a solution in wave gauge the constraint is $\mathcal{C}_A = 0$. The only solution tending to zero with x^1 is $\nu_A \equiv 0$.

5. $\mathcal{C}_0$ constraint

It holds that, in our coordinates,

$$\ell^\alpha\bar{S}_{0\alpha} \equiv \bar{S}_{01} \equiv \bar{R}_{01} - \frac{1}{2}\bar{g}_{01}\bar{R} \equiv -\frac{1}{2}\bar{g}^{AB}\bar{R}_{AB}.$$

A straightforward computation gives

$$\bar{S}_{01} \equiv \mathcal{C}_0 + \mathcal{L}_0, \quad \text{with } \mathcal{L}_0 \equiv -\partial_1\bar{H}^1 - \frac{n-1}{x^1}\bar{H}^1,$$
$$\mathcal{C}_0 \equiv \partial^2_{11}\bar{g}_{00} + \frac{3}{2}\frac{n-1}{x^1}\partial_1\bar{g}_{00} + \frac{(n-1)(n-2)}{2(x^1)^2}(1+\bar{g}_{00}).$$

Vanishing of $\bar{T}_{01}$

Definition. A stress energy tensor T satisfies the rigid energy condition on a null manifold with generator ℓ if the property $\bar{T}(\ell,\ell) = 0$ implies $\bar{T}(\cdot,\ell) = 0$.

Lemma 5.1. *The stress energy tensor of a perfect fluid or of a scalar field satisfying the wave equation satisfy the rigid energy condition on a null cone.*

Proof. The stress energy tensor of a perfect fluid is

$$T_{\alpha\beta} \equiv \rho u_\alpha u_\beta + p(u_\alpha u_\beta + g_{\alpha\beta}), \quad u^\alpha u_\alpha = -1,$$

therefore

$$\bar{T}_{11} \equiv (\bar{\rho} + \bar{p})(\bar{u}_1)^2.$$

Hence $\bar{T}_{11} = 0$ implies either $\bar{u}_1 = 0$, i.e. $\bar{u}$ orthogonal to the null vector ℓ, impossible for a timelike vector, or $\bar{\rho} + \bar{p} = 0$, impossible when $\bar{\rho} \geqslant 0$, except if $\bar{p} = -\bar{\rho}$, possible only for classical fluids if $\bar{p} = -\bar{\rho} = 0$, i.e. vacuum.

For a scalar field

$$T_{\alpha\beta} \equiv \partial_\alpha \phi \partial_\beta \phi - \frac{1}{2} g_{\alpha\beta} \partial_\lambda \phi \partial^\lambda \phi, \quad \bar{T}_{11} = (\partial_1 \bar{\phi})^2.$$

So $\partial_1 \bar{\phi} = 0$ implies $\bar{\phi} = \phi(O)$, i.e. $\bar{\phi} =$ constant, hence $\phi =$ constant if solution of the wave equation, then $T = 0$. □

Lemma 5.2. *The stress energy tensor of an electromagnetic field F,*

$$T_{\alpha\beta} \equiv F^\lambda{}_\alpha F_{\lambda\beta} - \frac{1}{4} g_{\alpha\beta} F^{\lambda\mu} F_{\lambda\mu}$$

satisfies the rigid energy condition if $\nu_0 = 1$, $\nu_A = 0$ and F satisfies on C_O the Maxwell constraints.

Proof. The condition $\bar{T}_{11} = 0$ reads in our coordinates

$$\bar{T}_{11} \equiv \bar{g}^{\lambda\mu} \bar{F}_{\mu 1} \bar{F}_{\lambda 1} \equiv \bar{g}^{AB} \bar{F}_{A1} \bar{F}_{B1} = 0,$$

which implies $\bar{F}_{A1} = 0$, hence also $\bar{F}^{A0} = 0$, and $\bar{T}_{1A} = 0$ as foreseen since the Maxwell tensor is known to satisfy the dominant energy condition. We consider $\bar{T}_{01}$ under the hypothesis $\nu_0 = 1$, $\nu_A = 0$, $\bar{F}_{A1} = \bar{F}^{A0} = 0$, then after simplifications

$$\bar{T}_{01} = \frac{1}{2}(F_{01})^2 - \frac{1}{4} \bar{F}^{AB} \bar{F}_{AB}.$$

The Maxwell identity $\mathrm{d}F \equiv 0$ shows that

$$\partial_1 \bar{F}_{AB} = 0,$$

therefore $\bar{F}_{AB} = 0$ if it vanishes at the vertex O, which results for a smooth F from the polar character of the coordinates x^A.

The Maxwell equation $\mathcal{E} \equiv \delta F = 0$ induces on C_O the constraint

$$\ell^\beta \bar{\mathcal{E}}_\beta \equiv \bar{\mathcal{E}}_1 = 0.$$

Using previous results, this implies (s metric of round S^{n-1})

$$\partial_1 \left(\nu^0 \sqrt{\det \tilde{g}}\, \overline{F_{01}}\right) = \partial_1 \left\{ \left(x^1\right)^{n-2} \sqrt{\det s}\, \overline{F_{01}} \right\} = 0,$$

hence $(x^1)^{n-2} \sqrt{\det s}\, \overline{F_{01}}$ is constant on a ray, therefore zero on a ray if $\overline{F_{01}}$ is bounded near $x^1 = 0$. Thus $\overline{F_{01}} = 0$ and, finally, $\bar{T}_{01} = 0$. □

Solution of the constraint $\mathcal{C}_0 = 0$.

Classical results on Fuchsian ODE give the following theorem:

Lemma 5.3. *The only solution of the constraint $\mathcal{C}_0 = 0$ such that $\bar{g}_{00}$ tends to minus one as x^1 tends to zero is $\bar{g}_{00} = -1$.*

6. Light-cone theorem

Before proving that the only space–time with the Minkowskian initial data that we have obtained is Minkowskian in an appropriate region, we must prove that it is vacuum.

We set $C_s := C_O \cap \{0 \leqslant x^1 \leqslant s\}$ and denote by Y_t a domain in the future of C_t sliced by hypersurfaces Σ_s, $0 \leqslant s \leqslant t$, with timelike normals n, and boundary $C_t \cup \Sigma_t$.

Lemma 6.1. *If the stress energy tensor source of the Einstein equations satisfies the dominant and rigid energy conditions on C_t, then it vanishes in Y_t if $\bar{T}(\ell, \ell)$ vanishes on C_t.*

Proof. If g satisfies the Einstein equations (2) then $\nabla_\alpha T^{\alpha\beta} = 0$. Given a vector field X it holds therefore that

$$\nabla_\alpha \left(T^{\alpha\beta} X_\beta\right) = \frac{1}{2} L_X g.T. \tag{23}$$

Integration on Y_t and the Stokes formula give

$$\int\limits_{\Sigma_t} T(X,n)\omega_t \leqslant \int\limits_{C_t} \overline{T(X,\ell)}\omega_{C_t} + \int\limits_0^t\int\limits_{\Sigma_s} \frac{1}{2}|L_X g.T|\omega_s \, \mathrm{d}s,$$

where the ω_*'s denote the relevant volume forms. Choosing for X a timelike vector, the dominant energy condition satisfied by T implies that there exists a constant C such that $|L_X g.T| \leqslant CT(X,n)$. The Gronwall lemma shows then that $T(X,n) = 0$ if $\overline{T(X,\ell)} = 0$, hence $T = 0$ in Y_t. □

Remark. If g is the Minkowski metric η then the above proof shows, without the rigid dominant hypothesis, that a traceless stress energy tensor T is zero if it satisfies the dominant energy condition and $\bar{T}(\ell,\ell) = 0$ on C_t; this property is easy to prove by taking for X in Y_t the vector $X^\mu = y^\mu$, y^μ Cartesian coordinates of $\mathbb{R}^{n+1}$, which is timelike in the interior of Y_t, colinear with ℓ on C_O and such that $L_X\eta = 2\eta$. The same property can be proved for more general metrics.

We can now complete as follows Theorem 3.1 [2]:

Theorem 6.2. *When the source T satisfies the dominant energy condition and $\bar{T}$ satisfies the rigid energy condition on C_O, the area of the sections $x^1 = s$ of the characteristic cone C_O is equal to the corresponding area of a Minkowskian cone if and only if, in the relevant domain of dependence, $T \equiv 0$ and the space–time is Minkowskian there.*

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