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A Note on FBSDE characterization of mean exit times ☆

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Abstract

In this Note, we present a new explicit characterization for a mean exit time problem recently treated by the author, in form of a quadratic Forward–Backward Stochastic Differential Equation (FBSDE) with a random terminal time. An a priori estimate and a uniqueness result for such a type of FBSDE are also proved, under certain conditions. ***To cite this article: C. Makasu, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Caractérisation des temps de sortie moyens pour une équation FBSDE. Dans cette Note on donne une nouvelle caractérisation explicite des temps de sortie moyens pour un probléme récemment introduit par l'auteur ; cette caractérisation est obtenue à partir d'une FBSDE quadratique à temps terminal aléatoire. On démontre aussi, sous certaines conditions, une estimation a priori, et un résultat d'unicité pour ce type d'équation différentielle stochastique directe et rétrograde. ***Pour citer cet article : C. Makasu, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Version française abrégée

Dans cette Note, nous présentons, sous certaines conditions, une nouvelle caractérisation explicite du temps de sortie moyen $\mathbb{E}^{x,y}[\tau_{x,y}]$ pour un processus de diffusion couplé $Q_t = (x_t, y_t)$ à l'intérieur du domaine $D_d = \{(x, y) \in \mathbb{R}^2_+ \colon\ y > \mathrm{d}(x)\}$. Nous démontrons que le temps de sortie moyen $\mathbb{E}^{x,y}[\tau_{x,y}] = \log(y\mathbb{E}^x p_0)$, où (x_t, y_t, p_t, q_t) est l'unique solution d'une équation FBSDE quadratique, faiblement couplée :

$$x_t = x + \int_0^{t\wedge\tau} \theta(x_s)\,\mathrm{d}s + \int_0^{t\wedge\tau} \beta(x_s)\,\mathrm{d}B_s^1,$$

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$$y_t = y + \mu \int_0^{t\wedge\tau} y_s \, \mathrm{d}s + \int_0^{t\wedge\tau} (\gamma\sqrt{x_s} + \alpha) y_s \, \mathrm{d}B_s^2,$$

$$p_t = \frac{1}{\mathrm{d}(x_\tau)} + \int_{t\wedge\tau}^{\tau} \left\{ \left(1 + \mu - \frac{1}{2}(\gamma\sqrt{x_s} + \alpha)^2\right) p_s - \frac{1}{2} I_{p_s \neq 0} \frac{q_s^2}{p_s} \right\} \mathrm{d}s - \int_{t\wedge\tau}^{\tau} q_s \, \mathrm{d}B_s^1,$$

où $\tau := \tau_{x,y} = \inf\{t > 0{:}\ y_t = \mathrm{d}(x_t)\}$ est un temps final aléatoire.

1. Introduction

Nonlinear BSDEs were first introduced in the classical paper by Pardoux and Peng [15], under Lipschitz continuous conditions. Since then, there has been a lot of interest in the study of both BSDEs ([3–5,10–12,19]) and FBSDEs ([1,2,9,16,21,22]), under various assumptions. The motivation for the study of BSDEs and FBSDEs is their wider applications in mathematical finance/stochastic control ([6–8,17,20], etc.). In this Note, our main concern is to present a new explicit characterization of the mean exit time in [13] as a kind of quadratic FBSDE with random terminal time. An a priori estimate and a uniqueness result for such a type of FBSDE are also proved, under certain conditions.

Let $Q_t = (x_t, y_t)$ be a weakly coupled, non-degenerate diffusion process described by:

$$\begin{aligned} &\mathrm{d}x_t = \theta(x_t)\,\mathrm{d}t + \beta(x_t)\,\mathrm{d}B_t^1; \quad x(0) = x, \\ &\mathrm{d}y_t = \mu y_t\,\mathrm{d}t + (\gamma\sqrt{x_t} + \alpha) y_t\,\mathrm{d}B_t^2; \quad y(0) = y, \end{aligned} \tag{1}$$

initially starting at (x, y) in the interior of a curvilinear domain $D_d \subset \mathbb{R}_+^2$, given by,

$$D_d = \left\{(x, y) \in \mathbb{R}_+^2{:}\ y > \mathrm{d}(x)\right\}. \tag{2}$$

Let

$$\tau_{x,y} = \inf\left\{t > 0{:}\ y_t = \mathrm{d}(x_t)\right\} \tag{3}$$

be the first exit time for the process (x_t, y_t) from the domain D_d through $y = \mathrm{d}(x)$.

Here, μ, γ, α are some fixed constants, d(.) is a positive continuous, nondecreasing convex function such that $\mathrm{d}(0) \geqslant 0$, $\theta(.)$ and $\beta(.)$ are locally bounded Borel measurable functions, B_t^1 and B_t^2 are independent Brownian motions on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Throughout the Note, we shall impose the restriction that the process $Q_t = (x_t, y_t)$ evolves in the prescribed domain D_d under non-explosive conditions, see, for instance, Narita [14] and references given there.

2. Main results

It can be shown that, using the result in [13], the mean exit time $\mathbb{E}^{x,y}[\tau_{x,y}] = \log(y\varphi(x))$ is associated with the following second-order nonlinear ordinary differential equation:

$$\frac{1}{2}\beta^2(x)\varphi''(x) + \theta(x)\varphi'(x) + \left(\mu + 1 - \frac{1}{2}(\gamma\sqrt{x} + \alpha)^2\right)\varphi(x) = \frac{1}{2}\beta^2(x)\frac{\varphi'(x)^2}{\varphi(x)}, \tag{4}$$

on the open interval $(0, \infty)$, and subject to the boundary condition,

$$\varphi(x) = \frac{1}{\mathrm{d}(x)}. \tag{5}$$

At this point, it is natural and interesting to ask about the probabilistic interpretation (see also Peng [18]) of Eqs. (4) and (5), which is indeed the essence of the next assertion.

Lemma 2.1. *(Probabilistic interpretation.) Let x_t be an arbitrary diffusion process given in* (1) *for all $t \geqslant 0$. Suppose that $\varphi(.) \in C^2(0, \infty)$. Then, $\varphi(x)$ admits the probabilistic interpretation*

$$\varphi(x) = \mathbb{E}^x p_0, \tag{6}$$

where (x_t, p_t, q_t) solves the one-dimensional, weakly coupled quadratic FBSDE:

$$x_t = x + \int_0^{t\wedge\tau} \theta(x_s)\,\mathrm{d}s + \int_0^{t\wedge\tau} \beta(x_s)\,\mathrm{d}B_s^1,$$

$$p_t = \frac{1}{\mathrm{d}(x_\tau)} + \int_{t\wedge\tau}^{\tau} \left\{ \left(1+\mu-\frac{1}{2}(\gamma\sqrt{x_s}+\alpha)^2\right)p_s - \frac{1}{2} I_{p_s\neq 0}\frac{q_s^2}{p_s} \right\}\mathrm{d}s - \int_{t\wedge\tau}^{\tau} q_s\,\mathrm{d}B_s^1, \tag{7}$$

with a random terminal time $\tau := \tau_{x,y}$.

The main result of this Note is stated as follows:

Proposition 2.1. *Let D_d be a curvilinear domain given by* (2)*, and let $Q_t = (x_t, y_t)$ be a non-degenerate, coupled diffusion process given by* (1) *in the interior of the domain D_d. Suppose that conditions* (H4.1), (H4.2) *below hold and $\mathbb{E}^{x,y}[\tau_{x,y}] < \infty$ a.s., then the mean exit time $\Phi(x,y) := \mathbb{E}^{x,y}[\tau_{x,y}]$ is characterized explicitly by*:

$$\Phi(x,y) = \log\big(y\mathbb{E}^x p_0\big), \tag{8}$$

where (x_t, p_t, q_t) uniquely solves the FBSDE (7)*.*

For the rest of the Note, our main concern is to prove the uniqueness of solution for the BSDE in (8) and an a priori estimate of the FBSDE (7). For this reason, we need a precise definition of the solution for (7) as given below. We first introduce the following appropriate function spaces. Denote

$$\mathcal{U}^2 = \left\{ g(t,\omega)\colon\ g(t,\omega) \text{ is } \mathcal{F}_t\text{-adapted real-valued such that } \mathbb{E}\int_0^{\tau} g^2(s,\omega)\,\mathrm{d}s < \infty \right\},$$

similarly $\mathcal{V}^2$, and

$$L^2(\Omega, \mathcal{F}_\tau, \mathbb{P}) = \big\{\xi\colon\ \xi \text{ is an } \mathcal{F}_\tau\text{-measurable random variable such that } \mathbb{E}\xi^2 < \infty\big\},$$

where $\tau := \tau_{x,y}$ and is given by (3).

Definition 2.1. A triple of $\mathcal{F}_t$-adapted processes $(x(.), p(.), q(.))$ is said to be a solution of the FBSDE (7), iff $(x(.), p(.), q(.)) \in \mathcal{U}^2 \times \mathcal{U}^2 \times \mathcal{V}^2$ and it satisfies (7).

We shall assume the following:

(H4.1) $\theta(.)$ and $\beta(.)$ are measurable functions on $(0,\infty)$ and there exists $K > 0$ such that for $x, z \in (0,\infty)$,

$$\big(\theta(x)-\theta(z)\big)^2 + \big(\beta(x)-\beta(z)\big)^2 \leqslant K(x-z)^2, \quad \theta^2(x)+\beta^2(x) \leqslant K^2\big(1+x^2\big).$$

(H4.2) $\mathrm{d}(x_\tau) \in L^2(\Omega, \mathcal{F}_\tau, \mathbb{P})$ for each x, and $\mathrm{d}(.)$ is a positive continuous, nondecreasing convex function such that $\mathrm{d}(0) \geqslant 0$.

Lemma 2.2. *(A priori estimate.) Assume that* (H4.1), (H4.2) *hold and $\mathbb{E}[t\wedge\tau] < \infty$ for all $t \geqslant 0$. If $(x_t, p_t, q_t) \in \mathcal{U}^2 \times \mathcal{U}^2 \times \mathcal{V}^2$ is a solution of the FBSDE* (7)*, then*

$$\mathbb{E}\left\{ x_t^2 + p_t^2 - k_1\int_0^{t\wedge\tau} x_s^2\,\mathrm{d}s + \int_{t\wedge\tau}^{\tau}\big(k_2 - k_3 x_s^2\big)p_s^2\,\mathrm{d}s - 2\mathbb{E}\int_{t\wedge\tau}^{\tau} q_s^2\,\mathrm{d}s \right\} \leqslant x^2(0) + \mathbb{E}\left\{ \frac{1}{\mathrm{d}^2(x_\tau)} + K^2\int_0^{t\wedge\tau}\mathrm{d}s \right\},$$

for all $t \geqslant 0$, where $k_1 = 1+K^2$, $k_2 = 2(1+\mu) - 3\gamma^2/2 - \gamma\alpha$ and $k_3 = \gamma(\alpha+\gamma/2)$.

Lemma 2.3. *(Uniqueness of solution.) Assume that* (H4.1) *and* (H4.2) *hold, then the FBSDE* (7) *has at most one solution in* $\mathcal{U}^2 \times \mathcal{U}^2 \times \mathcal{V}^2$.

Proof. Let (x_t^i, p_t^i, q_t^i), $i = 1, 2$, be two solutions of (7). Denote $\widehat{X}_t = x_t^1 - x_t^2$, $\widehat{P}_t = p_t^1 - p_t^2$, and $\widehat{Q}_t = q_t^1 - q_t^2$. Assume that (H4.1) holds. Applying Ito's formula to $\widehat{X}_s e^{\lambda \widehat{P}_s^2}$ where $\lambda \in \mathbb{R}$, it follows that

$$\begin{aligned}
\mathbb{E}\{\widehat{X}_\tau e^{\lambda \widehat{P}_\tau^2}\} &= \widehat{X}(0)e^{\lambda \widehat{P}^2(0)} + \mathbb{E}\left\{\int_0^\tau \left(\theta(\widehat{X}_t) + 2\lambda \widehat{P}_t \widehat{Q}_t \beta(\widehat{X}_t) + 2\lambda \widehat{X}_t \widehat{Q}_t^2\right) e^{\lambda \widehat{P}_t^2}\,\mathrm{d}t\right\} \\
&\quad + \mathbb{E}\left\{\int_0^\tau \left(\lambda[\gamma^2 - 2(1+\mu)]\widehat{X}_t \widehat{P}_t^2 + 2\lambda^2 \widehat{X}_t \widehat{Q}_t^2 \widehat{P}_t^2 + \lambda\gamma^2 \widehat{X}_t^2 \widehat{P}_t^2\right) e^{\lambda \widehat{P}_t^2}\,\mathrm{d}t\right\} \\
&\quad + 2\lambda\gamma\alpha\mathbb{E}\int_0^\tau \widehat{X}_t^{3/2} \widehat{P}_t^2 e^{\lambda \widehat{P}_t^2}\,\mathrm{d}t \\
&\leqslant \widehat{X}(0)e^{\lambda \widehat{P}^2(0)} + \mathbb{E}\left\{\int_0^\tau \left(\frac{K}{2}\widehat{X}_t^2 + 2\lambda \widehat{P}_t \widehat{Q}_t \beta(\widehat{X}_t) + 2\lambda \widehat{X}_t \widehat{Q}_t^2\right) e^{\lambda \widehat{P}_t^2}\,\mathrm{d}t\right\} \\
&\quad + \mathbb{E}\left\{\int_0^\tau \left(\lambda[\gamma^2 - 2(1+\mu)]\widehat{X}_t \widehat{P}_t^2 + 2\lambda^2 \widehat{X}_t \widehat{Q}_t^2 \widehat{P}_t^2 + \lambda\gamma^2 \widehat{X}_t^2 \widehat{P}_t^2\right) e^{\lambda \widehat{P}_t^2}\,\mathrm{d}t\right\} \\
&\quad + 2\lambda\gamma\alpha\mathbb{E}\int_0^\tau \widehat{X}_t^{3/2} \widehat{P}_t^2 e^{\lambda \widehat{P}_t^2}\,\mathrm{d}t + \frac{1}{2}\mathbb{E}\int_0^\tau e^{\lambda \widehat{P}_t^2}\,\mathrm{d}t \\
&\leqslant \widehat{X}(0)e^{\lambda \widehat{P}^2(0)} + \frac{1}{2}\mathbb{E}\int_0^\tau e^{\lambda \widehat{P}_t^2}\,\mathrm{d}t + \mathbb{E}\left\{\int_0^\tau \left(\lambda \widehat{Q}_t^2 + \lambda(\lambda+1)\widehat{P}_t^2 \widehat{Q}_t^2 + \frac{\lambda\sigma_0}{2}\widehat{P}_t^2\right) e^{\lambda \widehat{P}_t^2}\,\mathrm{d}t\right\} \\
&\quad + \mathbb{E}\left\{\int_0^\tau \left(K\left(\frac{1}{2} + \lambda\right) + \lambda\left(\frac{\sigma_0}{2} + \frac{3\gamma\alpha}{2} + \gamma^2\right)\widehat{P}_t^2 + \lambda \widehat{Q}_t^2 + \lambda^2 \widehat{Q}_t^2 \widehat{P}_t^2\right) \widehat{X}_t^2 e^{\lambda \widehat{P}_t^2}\,\mathrm{d}t\right\},
\end{aligned}$$

where the inequalities follow from using condition (H4.1) and Young's inequality, and where $\sigma_0 = \gamma^2 - 2(1+\mu) + \gamma\alpha$.

From the uniqueness of the FSDE in (1) and by choice of $-1 \leqslant \lambda < 0$ and $\sigma_0 > 0$, we deduce that the third term in the right-hand side of the last inequality is negative. Hence, $p_t^1 = p_t^2$ and $q_t^1 = q_t^2$ for all $t \in [0, \tau]$. This completes the proof. □

Example 2.1. Let D_d and $\tau_{x,y}$ be given by (2) and (3) respectively, where $\mathrm{d}(x) = e^x$. Consider a bidimensional geometric Brownian motion $Q_t = (x_t, y_t)$ in the interior of the domain D_d, given by:

$$\begin{aligned}
&\mathrm{d}x_t = \theta_0 x_t\,\mathrm{d}t + \beta_0 x_t\,\mathrm{d}B_t^1; \quad x(0) = x, \\
&\mathrm{d}y_t = \mu y_t\,\mathrm{d}t + \alpha y_t\,\mathrm{d}B_t^2; \quad y(0) = y,
\end{aligned} \tag{9}$$

where θ_0, β_0, μ and α are some fixed positive constants.

Our main result asserts that, for the above example, the mean exit time $\mathbb{E}^{x,y}[\tau_{x,y}]$ is characterized explicitly by

$$\mathbb{E}^{x,y}[\tau_{x,y}] = \log\left(y\mathbb{E}^x p_0\right),$$

where (x_t, p_t, q_t) uniquely solves the weakly, coupled quadratic FBSDE:

$$x_t = x + \theta_0 \int_0^{t\wedge\tau} x_s\,\mathrm{d}s + \beta_0 \int_0^{t\wedge\tau} x_s\,\mathrm{d}B_s^1,$$

$$p_t = e^{-x_\tau} + \int_{t\wedge\tau}^{\tau} \left\{ \left(1 + \mu - \frac{1}{2}\alpha^2\right) p_s - \frac{1}{2} I_{p_s \neq 0} \frac{q_s^2}{p_s} \right\} \mathrm{d}s - \int_{t\wedge\tau}^{\tau} q_s \, \mathrm{d}B_s^1,$$

with a random terminal time $\tau := \tau_{x,y} = \inf\{t > 0\colon\ y_t = e^{x_t}\}$.

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