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Algebra

On chains in division algebras of degree 3

Darrell Haile [a], Jung-Miao Kuo [b], Jean-Pierre Tignol [c,1]

[a] *Department of Mathematics, Indiana University, Bloomington, IN 47401, USA*
[b] *Department of Mathematics, National Taiwan University, Taipei, Taiwan*
[c] *Université catholique de Louvain, chemin du cyclotron, 2, B-1348 Louvain-la-Neuve, Belgium*

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Abstract

Let D be a division algebra of degree 3 over a field containing a primitive cube root of unity. We give two proofs of a theorem of Rost asserting that any two Kummer elements in D can be connected by a chain of length 4. ***To cite this article: D. Haile et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Chaines d'éléments de Kummer dans les corps gauches de degré 3. Soit D un corps gauche de degré 3 sur un corps contenant une racine cubique de l'unité. Nous donnons deux démonstrations d'un théorème de Rost établissant que deux éléments de Kummer quelconques de D peuvent être joints par une chaine de longueur 4. ***Pour citer cet article : D. Haile et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

1. Introduction

Let F be a field containing a primitive cube root of unity, ω. Let D be an F-central division algebra of degree 3. An element $x \in D$ is called *Kummer* if $x^3 \in F^\times$, but $x \notin F$. If x and y are Kummer we say the pair (x, y) is an *ω-pair* if $yx = \omega xy$. We also denote this by an arrow from x to y, so $x \to y$ means (x, y) is an ω-pair. By a well known theorem of Wedderburn every F-central division algebra of degree 3 contains an ω-pair. If x and y are Kummer elements in D, a *chain* from x to y is a finite sequence $x = x_0 \to x_1 \to x_2 \to \cdots \to x_n = y$. We call the integer n the *length* of the chain.

The question of whether, given Kummer elements x and y, there is a finite chain from x to y was considered by Rost [4] who proved:

E-mail addresses: haile@indiana.edu (D. Haile), jungmiaokuo@gmail.com (J.-M. Kuo), jean-pierre.tignol@uclouvain.be (J.-P. Tignol).
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Theorem 1.1 *(Rost). Any two Kummer elements $x, y \in D$ are connected by a chain of length* 4: *there exist Kummer elements z_1, z_2, z_3 in D such that $x \to z_1 \to z_2 \to z_3 \to y$.*

The purpose of this Note is to present two new proofs of this result. Both proofs are different from that of Rost and are strikingly different from each other; the first is entirely geometric and the second is quite explicit and computational.

2. Geometric proof of Rost's theorem

The geometry we refer to is that provided by the coefficients of the reduced characteristic polynomial of elements in D. For $x \in D$, let $t^3 - \mathrm{Tr}(x)t^2 + \mathrm{Sr}(x)t - \mathrm{Nr}(x) \in F[t]$ be the reduced characteristic polynomial of x, so Tr is the reduced trace. Let D_0 denote the set of elements of reduced trace 0, a subspace of D of dimension 8. The quadratic form Sr on D_0 has polar bilinear form $b(x, y) = -\mathrm{Tr}(xy)$ by [3, (34.14)], and it is hyperbolic: The algebra D can be split by an odd degree (in fact degree 3) extension of F and in the split case an easy computation shows that the Witt index of the form is 4. By Springer's theorem the form Sr on D therefore has Witt index 4 and so is hyperbolic on D_0.

The totally isotropic subspaces of D_0 form a geometry, an example of a so-called polar space. The axioms and basic properties of polar spaces can be found in Sections 7.3 and 7.4 of Cameron [2]. To emphasize the geometric point of view we consider more generally a (smooth) projective quadric X of dimension 6 defined by a hyperbolic quadratic form over F. The corresponding polar space, consisting of linear subspaces contained in X, has points, lines, and two types of linear spaces of dimension 3, which we call *solids*. We let $\mathfrak{P}$, $\mathfrak{L}$, $\mathfrak{S}_+$ and $\mathfrak{S}_-$ denote respectively the sets of points, lines, and solids of the two different types. The incidence relation between points and lines, and between lines and solids, is the inclusion. Two solids of different types are called *incident* if they intersect in a plane; otherwise, their intersection is a point. As usual, two points are called *collinear* if there is a line to which they are both incident. We also call solids of the same type collinear if there is a line that is incident to both of them, which amounts to saying their intersection is a line (unless they coincide); if they are not collinear, their intersection is empty. It should be said that one could prove our results in the setting of an abstract polar space of rank 4 which is "hyperbolic" in the sense that every linear subspace of dimension 2 is contained in exactly two solids. However such polar spaces were shown by Tits ([5, Theorem 7.12 and Proposition 8.4.3]) to be isomorphic to the one arising from a hyperbolic quadric as above and so there would be no real added generality.

Lemma 2.1. *Let $a, b \in \mathfrak{P}$, $S_+ \in \mathfrak{S}_+$, and $S_- \in \mathfrak{S}_-$. If S_+ and S_- are incident, there is a point $c \in \mathfrak{P}$ that is collinear with a and b, and incident to S_+ and S_-.*

Proof. By hypothesis, the intersection $S_+ \cap S_-$ is a plane. This intersection meets the intersection of the tangent hyperplanes to X at a and b in at least one point c, which meets all the requirements. □

The polar space X admits (geometric) triality, that is, there is a permutation

$$t: \mathfrak{P} \sqcup \mathfrak{L} \sqcup \mathfrak{S}_+ \sqcup \mathfrak{S}_- \to \mathfrak{P} \sqcup \mathfrak{L} \sqcup \mathfrak{S}_+ \sqcup \mathfrak{S}_- \quad \text{with} \quad t(\mathfrak{P}) = \mathfrak{S}_+, t(\mathfrak{S}_+) = \mathfrak{S}_-, t(\mathfrak{S}_-) = \mathfrak{P},$$

which preserves the incidence relations, and such that t^3 is the identity. The proof of the existence of such a triality map can be found in Section 8.6 of Cameron [2].

Any triality t yields a geometric version of ω-pair: For $a, b \in \mathfrak{P}$, we write $a \to b$ if the following two conditions hold: (i) a and b are collinear, and (ii) a and $t(b)$ are incident. Note that condition (ii) is also equivalent to: $t(a)$ and $t^2(b)$ are incident, or to: b and $t^{-1}(a)$ are incident.

Proposition 2.1. *If $a, b \in \mathfrak{P}$ are such that b is incident to $t(a)$, then there exists $c \in \mathfrak{P}$ such that $a \to c \to b$.*

Proof. The hypothesis implies that $t(b)$ and $t^{-1}(a)$ are incident. Therefore, Lemma 2.1 yields $c \in \mathfrak{P}$ that is collinear with a and b, and incident to $t(b)$ and $t^{-1}(a)$. □

Theorem 2.1. *For any $a, b \in \mathfrak{P}$, there exist c_1, c_2, $c_3 \in \mathfrak{P}$ such that $a \to c_1 \to c_2 \to c_3 \to b$.*

Proof. Since $t(a)$ and $t^{-1}(b)$ are solids of different types, their intersection is not empty. Let $c_2 \in \mathfrak{P}$ be incident to $t(a)$ and $t^{-1}(b)$. Then b is incident to $t(c_2)$, and Proposition 2.1 yields $c_1, c_3 \in \mathfrak{P}$ such that $a \to c_1 \to c_2$ and $c_2 \to c_3 \to b$. □

The length of chains connecting two points can be shortened in some cases.

Proposition 2.2. *If $a, b \in \mathfrak{P}$ are collinear, then there exist $c_1, c_2 \in \mathfrak{P}$ such that $a \to c_1 \to c_2 \to b$.*

Proof. Let $\ell \in \mathfrak{L}$ be incident to a and b. Then $t(a)$ and $t(b)$ are incident to $t(\ell)$. Let $c_2 \in \mathfrak{P}$ be the intersection of $t(\ell)$ with the hyperplane tangent to X at b. Then c_2 and b are collinear and c_2 is incident to $t(b)$, hence $c_2 \to b$. On the other hand, c_2 is incident to $t(a)$, hence Proposition 2.1 yields $c_1 \in \mathfrak{P}$ such that $a \to c_1 \to c_2$. □

We apply these results to the case of the form Sr on D_0. For this polar space X the points are the one-dimensional vector spaces spanned by Kummer elements of D_0, the lines and solids are the two-dimensional totally isotropic subspaces and the maximal (four-dimensional) totally isotropic subspaces of D_0, respectively, viewed projectively. Two points $a = xF$, $b = yF$ are collinear if they lie on the same line, which is equivalent to the condition $\mathrm{Tr}(xy) = 0$. Moreover one can write down an explicit triality map in terms of the Okubo product. We recall the definition: For $x, y \in D_0$ the Okubo product $x * y$ is given by

$$x * y = \frac{yx - \omega xy}{1 - \omega} - \frac{1}{3}\mathrm{Tr}(xy).$$

For each Kummer element x the spaces $x * D_0 = \{x * y \mid y \in D_0\}$ and $D_0 * x = \{y * x \mid y \in D_0\}$ are maximal totally isotropic subspaces of D_0 and these give the two kinds of solids. A triality map t is then given by $t: xF \mapsto x * D_0 \mapsto D_0 * x \mapsto xF$. For the details see Sections 34 and 35 of the Book of Involutions, Knus et al. [3]. In particular from Proposition 34.19 of [3], we have

$$x * (y * x) = (x * y) * x = -\frac{1}{3}\mathrm{Sr}(x)y \tag{1}$$

for all $x, y \in D_0$. It follows from this that if x is a Kummer element then $t(x) = x * D_0 = \{z \in D_0 \mid z * x = 0\}$. Therefore for points $a = xF$, $b = yF$ in this geometry the condition $a \to b$ (that is, a and b are collinear and a is incident to $t(b)$) is equivalent to $\mathrm{Tr}(xy) = 0$ and $x * y = 0$. From the definition of the product this is exactly the condition that (x, y) is an ω-pair (and so $xF \to yF$ if and only if $x \to y$). Rost's Theorem 1.1 thus readily follows from Theorem 2.1. Moreover, Proposition 2.2 yields:

Corollary 2.2. *Let D be an F-central division algebra of degree 3 and let x, y be Kummer elements in D. If xF and yF are collinear (that is, if $\mathrm{Tr}(xy) = 0$), there is a chain from x to y of length 3.*

Note that the geometric argument also applies in other cases where a geometric triality is defined, such as the space of symmetric elements of reduced trace 0 in a division algebra of degree 3 endowed with a distinguished unitary involution τ satisfying $\tau(\omega) = \omega^{-1}$, or the split octonion algebra (see [1, §3]).

3. Algebraic proof of Rost's theorem

Our second proof of Rost's theorem is noteworthy in that it produces an explicit formula for a chain of length 4 between two Kummer elements. As before let x, y be Kummer elements in the degree 3 algebra D. We will again use the Okubo product $*$, but also the product $\diamond$ defined on the space D_0 of trace zero elements by

$$x \diamond y = xy - \frac{1}{3}\mathrm{Tr}(xy).$$

Note that $x \diamond y = 0$ if and only if $xy \in F$, that is $y = \alpha x^2$, for some $\alpha \in F^\times$.

Lemma 3.1. *Let x, y be Kummer elements in D with $xy \notin F$. If $x * y = 0$, then we have a chain $x \to x \diamond y \to y$.*

Proof. Because $x * y = 0$, we have $yx - \omega xy = (\frac{1-\omega}{3})\,\mathrm{Tr}(xy)$. Hence

$$y(x \diamond y)y^{-1} = yx - \frac{1}{3}\,\mathrm{Tr}(xy) = \omega xy - \frac{\omega}{3}\,\mathrm{Tr}(xy) = \omega x \diamond y.$$

A similar calculation shows $(x \diamond y)x = \omega x(x \diamond y)$. Since $xy \notin F$, it follows that $x \diamond y \neq 0$ and so we have the desired chain. $\square$

We break the proof of Rost's Theorem 1.1 into several cases. We will produce chains of lengths at *most* 4 so we first observe that the length of any chain can be increased: If (x, y) is an ω-pair, then $x \to xy \to y$ is a chain of length 2.

We assume first that $xy \in F$, so $y = \alpha x^2$ for some $\alpha \in F^\times$. In that case if v is any element of D_0 such that $vx = \omega xv$, then we have the chain $x \to v \to y$. Also if $y = \alpha x$ for some $\alpha \in F$, then we have the chain $x \to v \to x^2 \to v^2 \to y$.

So we may assume y is not a multiple of x or x^2. If $y * x = 0$ then the previous lemma produces a chain $x \to (y \diamond x)^2 \to y$ of length 2.

Now we assume $y * x \neq 0$ and y is not a multiple of x or x^2. By (1), we have $x * y * x = y * x * y = 0$. Assume $x(y * x) \in F$. Then as above we have a chain $x \to v \to y * x$ where $v \in D_0$ is any element such that $vx = \omega xv$. Because y is not a multiple of x or x^2, we must have $(y * x)y \notin F$ and so from the lemma we obtain a chain $y * x \to (y * x) \diamond y \to y$. Putting these two chains together gives the chain $x \to v \to y * x \to (y * x) \diamond y \to y$. A similar argument handles the case $(y * x)y \in F$.

Finally if we assume $y * x \neq 0$, y is not a multiple of x or x^2, and neither $x(y * x)$ or $(y * x)y$ is in F, then from $x * y * x = y * x * y = 0$ and the lemma we obtain the chain $x \to x \diamond (y * x) \to y * x \to (y * x) \diamond y \to y$. $\square$

To see that this last chain is not a chain one would guess easily, note that

$$x \diamond (y * x) = \frac{xyx - \omega x^2 y}{1 - \omega} - \frac{1}{3}\,\mathrm{Tr}(xy)x - \frac{1}{3}\,\mathrm{Tr}(xyx) - \frac{1}{3}\,\mathrm{Tr}(xy).$$

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