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On a residual local projection method for the Darcy equation ☆

Leopoldo P. Franca [a,b], Christopher Harder [a], Frédéric Valentin [c]

[a] University of Colorado Denver, P.O. Box 17364, Campus Box 170, Denver, CO 80217-3364, USA
[b] Department of Civil Engineering COPPE/UFRJ, P.O. Box 68506, 21945-970 Rio de Janeiro - RJ, Brazil
[c] Laboratório Nacional de Computação Científica (LNCC), Av. Getúlio Vargas, 333, 25651-070 Petrópolis - RJ, Brazil

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Abstract

A new symmetric local projection method built on residual bases (RELP) makes linear equal-order finite element pairs stable for the Darcy problem. The derivation is performed inside a Petrov–Galerkin enriching space approach (PGEM) which indicates parameter-free terms to be added to the Galerkin method without compromising consistency. Velocity and pressure spaces are augmented using solutions of residual dependent local Darcy problems obtained after a static condensation procedure. We prove the method achieves error optimality and indicates a way to recover a locally mass conservative velocity field. Numerical experiments validate theory. ***To cite this article: L.P. Franca et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Une méthode de projection local résiduelle pour l'équation de Darcy. On propose une nouvelle méthode de projection locale symétrique du type résiduel (RELP) pour l'équation de Darcy. La méthode est construite dans un cadre d'enrichissement des espaces d'interpolations par une approche du type Petrov–Galerkin, ce qui nous permet de modifier de façon naturelle la méthode de Galerkin et d'éviter le choix des constantes de stabilisation. L'approche d'enrichissement est basée sur la résolution de problèmes de Darcy locaux, qui dépendent des résidus après un procédé de condensation statique. On démontre que la méthode est stable pour les paires d'éléments finis linéaires continus et discontinus en pression. On établit, ensuite, l'optimalité de l'erreur dans les normes naturelles et on propose une stratégie de reconstruction de champ de vitesse localement conservatif. Les aspects théoriques sont validés numériquement. ***Pour citer cet article : L.P. Franca et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Version française abrégée

On adapte la stratégie d'enrichissement des espaces classiques d'éléments finis proposée dans [1] pour introduire une nouvelle méthode d'éléments finis du type projection locale (4). La méthode préserve les bonnes propriétés des méthodes de projection locales originales [4], comme la symétrie et l'indépendance par rapport aux constantes de

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E-mail addresses: leopoldo.franca@ucdenver.edu (L.P. Franca), christopher.harder@ucdenver.edu (C. Harder), valentin@lncc.br (F. Valentin).

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stabilisation. Cependant, différemment de [4] et [1], la méthode proposée dans ce travail est basée sur des aspects résiduels, et donc, la méthode est naturellement consistante. Cette propriété est obtenue en cherchant la solution dans des espaces d'éléments finis contenant les solutions des problèmes de Darcy locaux, à savoir, les problèmes (11)–(13). On établit l'existence et l'unicité de solution en démontrant une condition du type inf-sup dans le Lemme 1. Ensuite, on démontre que les erreurs d'approximation sont optimales dans les normes naturelles dans le Théorème 2, en utilisant fortement la propriété de consistance. Le Théorème 2 propose également une façon de reconstruire un champ de vitesse localement conservatif. Les tests numériques (Fig. 1 et Tableau 1) valident la théorie.

1. Introduction

Usually adopted in its mixed form, the Darcy problem is the basic tool in the study of fluid flow simulations in porous media. However, this mixed characterization is unstable when the Darcy problem is solved with equal-order linear interpolation via the Galerkin method [2]. Recently, Bochev and Dohrmann [4] stabilized such a pair for the Stokes problem with the addition to the Galerkin method of a symmetric term based on the difference between the pressure and its local projection onto polynomial spaces of one degree lower than the pressure space. Unlike most stabilized approaches, no parameter needs to be fixed. This comes at the price of relaxing consistency, although such error is shown to be at order of leading errors. A related idea in [3] recovers stability for the Darcy model by controlling the gradient of the pressure fluctuation instead. This version demands a parameter to be fixed and prevents the use of low order interpolation spaces. Moreover, the author noted a sub-optimal convergence in pressure due to the lack of consistency. This work introduces a new stable, symmetric, consistent, and constant-free method for the Darcy problem which overcomes drawbacks stated in [3]. The framework is of the Petrov–Galerkin type and follows guidelines presented in [1] for the Stokes equation, namely, finite element spaces enhanced with the solution of local problems which give rise to residual based terms that augment the Galerkin method and include the local pressure projections proposed in [4]. The method, named RELP, is based on a continuous linear space for the velocity and both discontinuous and continuous linear spaces for the pressure. As for the latter pair, the method is proved to be locally mass conservative. Classical approaches, such as the Galerkin method based on the Raviart–Thomas element or the finite volume methods, are also conservative. Practitioners will find the new RELP method to be advantageous as it achieves second order convergence of variables, may be easily incorporated into existing nodal-based codes, and ultimately leads to linear algebraic systems of smaller dimension. The outline of this work is as follows: this section ends with the problem statement and notations. Section 2 highlights the RELP method which is then derived in Section 3. Section 4 is devoted to stability and convergence results which are numerically validated in Section 5.

1.1. Preliminaries

Let Ω denote an open, bounded domain in $\mathbb{R}^2$ with polygonal boundary $\partial\Omega$. We seek the velocity and pressure solution $(\boldsymbol{u}, p)$ to the following mixed form of the Darcy problem:

$$\sigma \boldsymbol{u} + \nabla p = \boldsymbol{0}, \quad \nabla \cdot \boldsymbol{u} = g \quad \text{in } \Omega, \tag{1}$$

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0 \quad \text{on } \partial\Omega, \tag{2}$$

where $\sigma = \frac{\mu}{\kappa} \in \mathbb{R}^+$ in Ω, with μ and κ denoting the viscosity and permeability, respectively, and g is a given source such that $\int_\Omega g = 0$.

We adopt the usual definitions for the spaces $L^2(D)$, $L^2_0(D)$, $H(\mathrm{div}, D)$, and $H_0(\mathrm{div}, D)$ equipped with norms $\|.\|_{0,D}$ and $\|.\|_{\mathrm{div},D}$, respectively, where D is a bounded set. The symmetric weak formulation of problem (1)–(2) reads: *Find* $(\boldsymbol{u}, p) \in H_0(\mathrm{div}, \Omega) \times L^2_0(\Omega)$ such that

$$\mathbf{B}\big((\boldsymbol{u}, p), (\boldsymbol{v}, q)\big) = -(g, q)_\Omega \quad \text{for all } (\boldsymbol{v}, q) \in H_0(\mathrm{div}, \Omega) \times L^2_0(\Omega), \tag{3}$$

where $\mathbf{B}((\boldsymbol{u}, p), (\boldsymbol{v}, q)) := (\sigma \boldsymbol{u}, \boldsymbol{v})_\Omega - (\nabla \cdot \boldsymbol{v}, p)_\Omega - (\nabla \cdot \boldsymbol{u}, q)_\Omega$, and $(.,.)_D$ stands for the inner-product in $L^2(D)$. The problem (3) is then well-posed by supposing $g \in L^2_0(\Omega)$ (cf. [2]).

Next, we introduce a family $\mathcal{T}_h$ of regular partitions of $\overline{\Omega}$ composed of triangles K with boundary ∂K consisting of edges F. The set of internal edges of $\mathcal{T}_h$ is $\mathcal{E}_h$. The diameters of K and F read h_K and h_F, respectively, and $h := \max\{h_K \colon K \in \mathcal{T}_h\}$. Also, for each $F = K \cap K' \in \mathcal{E}_h$ we choose a fixed unit normal vector $\boldsymbol{n}_F$. The standard outward normal vector at the edge F with respect to the element K is denoted by $\boldsymbol{n}_F^K$, and coincides with $\boldsymbol{n}_F$ in the case $F \subset \partial\Omega$. Moreover, for a scalar function q, one denotes its jump as the vectorial quantity $[\![q]\!] := q|_K \boldsymbol{n}_F^K + q|_{K'} \boldsymbol{n}_F^{K'}$, and its projection by $\Pi_D(q) := \frac{1}{|D|}\int_D q$.

In what follows, V_1 stands for the finite element space of continuous, piecewise linear polynomials and we set $\mathbf{V}_1 := [V_1]^2 \cap H_0(\mathrm{div}, \Omega)$. Also, Q_1 is the space of piecewise linear polynomials which are continuous or discontinuous over Ω and belong to $L_0^2(\Omega)$. Denoting residuals of the momentum and mass equations, respectively, by $R^M(\boldsymbol{v}_1, q_1) := -\sigma \boldsymbol{v}_1 - \nabla q_1$ and $R^C(g, \boldsymbol{v}_1) := g - \nabla\cdot \boldsymbol{v}_1$, for $(\boldsymbol{v}_1, q_1)$ in $\mathbf{V}_1 \times Q_1$, we define the following finite-dimensional space:

$$W := \left\{q \in H^2(\mathcal{T}_h) \cap L_0^2(\mathcal{T}_h)\text{: } \Delta q = \nabla\cdot R^M(\boldsymbol{v}_1, q_1) \text{ in } K,\ \nabla q \cdot \boldsymbol{n}_F = \Pi_F\big(R^M(\boldsymbol{v}_1, q_1)\big)\cdot \boldsymbol{n}_F \text{ on } F \subset \partial K\right\},$$

and its orthogonal complement in $L_0^2(\mathcal{T}_h)$, denoted by $W^\perp$. From its definition one can prove that $W^\perp \cap Q_1 = \{0\}$. Here we use $H_0^1(\mathcal{T}_h) := \oplus\sum_{K\in\mathcal{T}_h} H_0^1(K)$ and $L_0^2(\mathcal{T}_h) := \oplus\sum_{K\in\mathcal{T}_h} L_0^2(K)$, and $H_0(\mathrm{div}, \mathcal{T}_h) := \oplus\sum_{K\in\mathcal{T}_h} H_0(\mathrm{div}, K)$.

2. The Residual Local Projection method

The symmetric RELP method reads: *Find*$(\boldsymbol{u}_1, p_1) \in \mathbf{V}_1 \times Q_1$ such that

$$\mathbf{B}_s\big((\boldsymbol{u}_1, p_1), (\boldsymbol{v}_1, q_1)\big) = -\sum_{K\in\mathcal{T}_h} \big(\Pi_K(g), q_1\big)_K \quad \text{for all } (\boldsymbol{v}_1, q_1) \in \mathbf{V}_1 \times Q_1, \tag{4}$$

where

$$\begin{aligned}\mathbf{B}_s\big((\boldsymbol{u}_1, p_1), (\boldsymbol{v}_1, q_1)\big) = {}& \mathbf{B}\big((\pi(\boldsymbol{u}_1), p_1), \big(\pi(\boldsymbol{v}_1), q_1\big)\big) - \sum_{F\in\mathcal{E}_h} \frac{1}{h_F\sigma}\big(\Pi_F\big([\![p_1]\!]\big), \Pi_F\big([\![q_1]\!]\big)\big)_F \\ & - \sum_{K\in\mathcal{T}_h} \frac{1}{h_K^2\sigma}\big(\mathcal{N}_K(\boldsymbol{u}_1) - p_1 + \Pi_K(p_1), \mathcal{N}_K(\boldsymbol{v}_1) - q_1 + \Pi_K(q_1)\big)_K.\end{aligned} \tag{5}$$

The global functions $\pi(\cdot)$ and $\mathcal{N}(\cdot)$ are defined such that $\pi(\boldsymbol{v}_1)|_K = \pi_K(\boldsymbol{v}_1)$ and $\mathcal{N}(\boldsymbol{v}_1)|_K = \mathcal{N}_K(\boldsymbol{v}_1)$, with $(\pi_K(\boldsymbol{u}_1), \mathcal{N}_K(\boldsymbol{u}_1))$ given by

$$\pi_K(\boldsymbol{u}_1) = \sum_{F\subset\partial K} \Pi_F\big(\boldsymbol{u}_1\cdot \boldsymbol{n}_F^K\big)\boldsymbol{\varphi}_F^K \quad \text{and} \quad \mathcal{N}_K(\boldsymbol{u}_1) = \sum_{F\subset\partial K} \Pi_F\big(\boldsymbol{u}_1\cdot \boldsymbol{n}_F^K\big)\eta_F^K. \tag{6}$$

The basis $\boldsymbol{\varphi}_F^K = \frac{h_F}{2|K|}(\mathbf{x} - \mathbf{x}_F)$, where $\mathbf{x}_F$ is the node opposite to the edge F, generates the lowest order Raviart–Thomas space and η_F^K belongs to $L_0^2(K)$ such that $\nabla\eta_F^K = -\sigma\boldsymbol{\varphi}_F^K$.

Remark 1. Continuity of the normal component of $\boldsymbol{u}_1$ is shared by $\pi(\boldsymbol{u}_1)$, while the tangential is not.

3. Derivation of the method

We start by looking for the solution $(\boldsymbol{u}_h, q_h)$ decomposed into large and small scales, respectively, $(\boldsymbol{u}_1, q_1) \in \mathbf{V}_1 \times Q_1$ and $(\boldsymbol{u}_e, q_e) \in H_0(\mathrm{div}, \Omega) \times L_0^2(\mathcal{T}_h)$. Then, the Petrov–Galerkin enriched method reads: *Find*$(\boldsymbol{u}_h, p_h) \in [\mathbf{V}_1 + H_0(\mathrm{div}, \Omega)] \times [Q_1 + L_0^2(\mathcal{T}_h)]$ such that

$$\mathbf{B}\big((\boldsymbol{u}_h, p_h), (\boldsymbol{v}_h, q_h)\big) = -(g, q_h)_\Omega,$$

for all $(\boldsymbol{v}_h, q_h) := (\pi(\boldsymbol{v}_1) + \boldsymbol{v}_b, q_1 + q_b) \in [\pi(\mathbf{V}_1) \oplus H_0(\mathrm{div}, \mathcal{T}_h)] \times [Q_1 \oplus W^\perp]$, where $\pi(\mathbf{V}_1)$ represents the space generated by applying operator $\pi(\cdot)$ on functions in $\mathbf{V}_1$. Now, using $\boldsymbol{v}_b \cdot \boldsymbol{n}|_{\partial K} = 0$ and $p_e|_K \in L_0^2(K)$, the problem above is equivalent to the following system:

$$\mathbf{B}\big((\boldsymbol{u}_1, p_1), \big(\pi(\boldsymbol{v}_1), q_1\big)\big) + \sum_{K\in\mathcal{T}_h} \big[\big(\sigma\boldsymbol{u}_e, \pi(\boldsymbol{v}_1)\big)_K - (\nabla\cdot\boldsymbol{u}_e, q_1)_K\big] = -(g, q_1)_\Omega, \tag{7}$$

$$(\sigma\boldsymbol{u}_e + \nabla p_e, \boldsymbol{v}_b)_K + (\sigma\boldsymbol{u}_1 + \nabla p_1, \boldsymbol{v}_b)_K - (\nabla\cdot\boldsymbol{u}_e, q_b)_K - (\nabla\cdot\boldsymbol{u}_1, q_b)_K = -(g, q_b)_K. \tag{8}$$

Next, denoting from now on $R^M = R^M(\boldsymbol{u}_1, p_1)$ and $R^C = R^C(g, \boldsymbol{u}_1)$, the weak problem (8) is equivalent to

$$\sigma \boldsymbol{u}_e + \nabla p_e = R^M, \quad \nabla\cdot\boldsymbol{u}_e - R^C \in W \oplus \mathbb{P}_0(K) \quad \text{in each } K, \tag{9}$$

where $\mathbb{P}_0(K)$ denotes the piecewise constant polynomial space.

We next choose boundary conditions with the intention of correcting the residual on the edges and keeping the approach conforming. To this end, we fix

$$\boldsymbol{u}_e\cdot\boldsymbol{n}_F = \left[R^M - \Pi_F\big(R^M\big) + \frac{1}{h_F\sigma}\Pi_F\big([\![p_1]\!]\big)\right]\cdot\boldsymbol{n}_F \quad \text{on } F\in\mathcal{E}_h, \tag{10}$$

and $\boldsymbol{u}_e\cdot\boldsymbol{n}_F = 0$ elsewhere.

It comes from (9) that $(\boldsymbol{u}_e, p_e)$ inherits the degrees of freedom of $(\boldsymbol{u}_1, p_1)$ and that it may be split into $(\boldsymbol{u}_e, p_e) = (\boldsymbol{u}_e^M, p_e^M) + (\boldsymbol{u}_e^G, p_e^G) + (\boldsymbol{u}_e^D, p_e^D)$, where each contribution satisfies, respectively,

$$\sigma\boldsymbol{u}_e^M + \nabla p_e^M = R^M, \quad \nabla\cdot\boldsymbol{u}_e^M = 0 \quad \text{in } K, \tag{11}$$
$$\sigma\boldsymbol{u}_e^M\cdot\boldsymbol{n}_F = \big(R^M - \Pi_F\big(R^M\big)\big)\cdot\boldsymbol{n}_F \quad \text{on } F\subset\partial K,$$
$$\sigma\boldsymbol{u}_e^G + \nabla p_e^G = \boldsymbol{0}, \quad \nabla\cdot\boldsymbol{u}_e^G - R^C \in W\oplus\mathbb{P}_0(K) \quad \text{in } K, \tag{12}$$
$$\boldsymbol{u}_e^G\cdot\boldsymbol{n}_F = 0 \quad \text{on} F\subset\partial K,$$

and

$$\sigma\boldsymbol{u}_e^D + \nabla p_e^D = \boldsymbol{0}, \quad \nabla\cdot\boldsymbol{u}_e^D \in \mathbb{P}_0(K) \quad \text{in } K, \tag{13}$$
$$\boldsymbol{u}_e^D\cdot\boldsymbol{n}_F = \frac{1}{h_F\sigma}\Pi_F\big([\![p_1]\!]\big)\cdot\boldsymbol{n}_F \quad \text{on } F\subset\partial K.$$

It remains to set (12) and (13). To this end, we first remark that the solution of (11) reads $\boldsymbol{u}_e^M = -\boldsymbol{u}_1 + \pi(\boldsymbol{u}_1)$ and $p_e^M = p_e^M(R^M) \in W$ is given by

$$p_e^M\big|_K = -p_1 + \Pi_K(p_1) + \mathcal{N}_K(\boldsymbol{u}_1). \tag{14}$$

Hence, we reinforce the dependence of the enriching functions in terms of residuals and preserve the compatibility condition by closing (12) and (13) as follows:

$$\nabla\cdot\boldsymbol{u}_e^G = R^C - \Pi_K\big(R^C\big) - \frac{1}{h_K^2\,\sigma}p_e^M, \quad \nabla\cdot\boldsymbol{u}_e^D = \frac{1}{|K|\sigma}\sum_{F\subset\partial K}\Pi_F\big([\![p_1]\!]\big)\cdot\boldsymbol{n}_F^K, \tag{15}$$

where we have used a standard dimensional argument to balance p_e^M.

We can now state the RELP method corresponding to solving (7) with $(\boldsymbol{u}_e, p_e)$ given by (11)–(13) and (15). Using (11), integrating by parts, and (10), the method reads: *Find*$(\boldsymbol{u}_1, p_1) \in \mathbf{V}_1 \times Q_1$ such that

$$\begin{aligned}
&\mathbf{B}\big(\big(\pi(\boldsymbol{u}_1), p_1\big), \big(\pi(\boldsymbol{v}_1), q_1\big)\big) + \sum_{K\in\mathcal{T}_h}\frac{1}{h_K^2\sigma}\big(\mathcal{N}_K(\boldsymbol{u}_1) - p_1 + \Pi_K(p_1), -\mathcal{N}_K(\boldsymbol{v}_1) + q_1\big)_K \\
&\quad + \sum_{K\in\mathcal{T}_h}\big(\boldsymbol{u}_e^D, \sigma\pi(\boldsymbol{v}_1) + \nabla q_1\big)_K - \sum_{F\in\mathcal{E}_h}\frac{1}{h_F\sigma}\big(\Pi_F\big([\![p_1]\!]\big), \Pi_F\big([\![q_1]\!]\big)\big)_F \\
&= -\sum_{K\in\mathcal{T}_h}\big[\big(\Pi_K(g), q_1\big)_K + \big(\sigma\boldsymbol{u}_e^G(g), \pi_K(\boldsymbol{v}_1)\big)_K\big],
\end{aligned}$$

where $\boldsymbol{u}_e^G(g)$ is the solution of (12)–(15) related to g. The final step consists of remarking that, as $\|\eta_F\|_{0,K}$ is of order h_K^2, the term $\sum_{K\in\mathcal{T}_h}(\sigma\boldsymbol{u}_e^G(g), \pi(\boldsymbol{v}_1))_K$ is of small size compared to the leading error, and the term $\sum_{K\in\mathcal{T}_h}(\boldsymbol{u}_e^D, \pi(\boldsymbol{v}_1) + \nabla q_1)_K$ may be handled as in [1], and then neglected without compromising optimality. This results in the symmetric method (4).

4. Error analysis

We define the following mesh dependent norm:

$$|||(\boldsymbol{u}, p)|||_h = \left[\sigma \|\boldsymbol{u}\|_{0,\Omega}^2 + \sum_{K \in \mathcal{T}_h} \frac{1}{\sigma} \|\nabla p\|_{0,K}^2 + \sum_{F \in \mathcal{E}_h} \frac{1}{\sigma h_F} \|\Pi_F([\![p]\!])\|_{0,F}^2\right]^{1/2}. \tag{16}$$

Being residual-based, the method (4) is immediately consistent. Uniqueness for the RELP method (4) holds from the following result:

Lemma 1. *Let $\mathbf{B}_s(.,.)$ be the bilinear form in* (5). *Then there exists a positive constant β, independent of h and σ, such that*

$$\sup_{(\boldsymbol{w}_1, r_1) \in \mathbf{V}_1 \times Q_1 - \{\mathbf{0}\}} \frac{\mathbf{B}_s((\boldsymbol{v}_1, q_1), (\boldsymbol{w}_1, r_1))}{|||(\pi(\boldsymbol{w}_1), r_1)|||_h} \geqslant \beta |||(\pi(\boldsymbol{v}_1), q_1)|||_h$$

for all $(\boldsymbol{v}_1, q_1) \in \mathbf{V}_1 \times Q_1$.

Standard interpolation results, Lemma 1, and consistency yield the following error estimate:

Theorem 2. *Let $(\boldsymbol{u}, p) \in [H^1(\Omega)^2 \cap H_0(\mathrm{div}, \Omega)] \times [H^2(\Omega) \cap L_0^2(\Omega)]$ be the solution of* (3) *and $(\pi(\boldsymbol{u}_1), p_1)$ be the solution of method* (4). *Then, there exists a positive constant C, independent of h and σ, such that*

$$|||(\boldsymbol{u} - \pi(\boldsymbol{u}_1), p - p_1)|||_h \leqslant Ch\left(\sqrt{\sigma}|\boldsymbol{u}|_{1,\Omega} + \frac{1}{\sqrt{\sigma}}|p|_{2,\Omega}\right),$$

$$\|p - p_1\|_{0,\Omega} \leqslant Ch^2\left(\sqrt{\sigma}|\boldsymbol{u}|_{1,\Omega} + \frac{1}{\sqrt{\sigma}}|p|_{2,\Omega}\right).$$

If the space Q_1 is discontinuous and $\boldsymbol{u} \in [H^2(\Omega)^2 \cap H_0(\mathrm{div}, \Omega)]$, we further have

$$\|\boldsymbol{u} - \pi(\boldsymbol{u}_1)\|_{\mathrm{div},\Omega} \leqslant Ch\left(|\boldsymbol{u}|_{2,\Omega} + \frac{1}{\sigma}|p|_{2,\Omega}\right),$$

and for all $K \in \mathcal{T}_h$

$$\int_K \nabla \cdot (\boldsymbol{u}_1 + \boldsymbol{u}_e^D) = \int_K g.$$

5. Numerical experiments

We set $\Omega = (0, 1) \times (0, 1)$, $\sigma = 1$, and we give the exact pressure $p = \sin(2\pi x)\sin(2\pi y)$. The velocity is computed from the Darcy's law, g is obtained from the divergence of velocity, and the boundary condition b is taken to be its normal component on the boundary. Fig. 1 depicts error curves for the discontinuous pressure case which are in agreement with theory. In addition, we observe optimal convergence in the L^2 norm for $\boldsymbol{u}_1$. Mimicking [4] in the context of Darcy model, i.e., stabilizing Galerkin method through a pressure projection only, convergence is lost. It seems that consistency is a necessary condition to recover convergence (see Fig. 1). Similar conclusions arise in the continuous case as well. Finally, Table 1 verifies the local mass conservation property.

Table 1
Mass conservation property for discontinuous pressure interpolation.

h	1/4	1/8	1/16	1/32	1/64
$\max_{K \in \mathcal{T}_h} \frac{\int_K g - \nabla \cdot (\boldsymbol{u}_1 + \boldsymbol{u}_e^D)}{\lvert K \rvert}$	6.2×10^{-12}	4.3×10^{-11}	1.9×10^{-10}	7.9×10^{-10}	3.3×10^{-9}

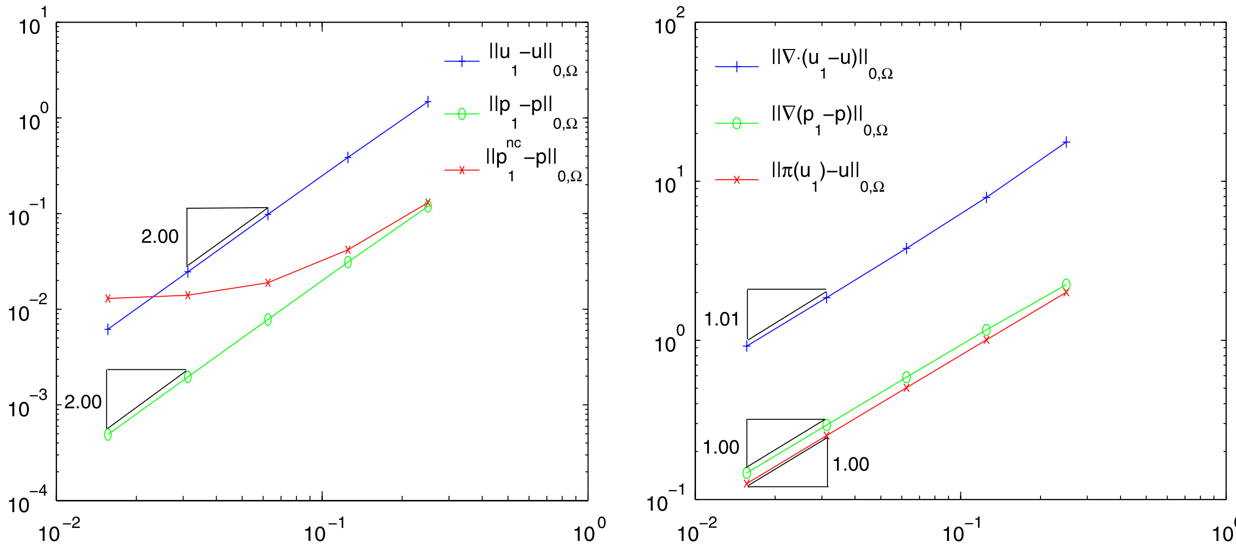

Fig. 1. Convergence study with discontinuous pressure. The pressure p_1^{nc} is calculated using the non-consistent method (or in another words, $\mathcal{N}_K(.)$ contribution is disregarded in (4)).

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