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Differential Geometry

The Lichnerowicz Laplacian on compact symmetric spaces

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Abstract

We show that, on a compact symmetric space, the Lichnerowicz Laplacian acting on the space of covariant tensor fields coincides with the Casimir operator and we deduce that, on a compact semisimple Lie group, the Lichnerowicz Laplacian is the mean of the left invariant Casimir operator and the right invariant Casimir operator. ***To cite this article: M. Boucetta, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Le laplacien de Lichnerowicz sur les espaces symétriques compacts. On montre que, dans un espace symétrique compact, le laplacien de Lichnerowicz agissant sur l'espace des tenseurs covariants coincide avec l'opérateur de Casimir et on déduit que, dans un groupe de Lie compact semisimple, le laplacien de Lichnerowicz est la moyenne de l'opérateur de Casimir invariant à gauche et celui invariant à droite. ***Pour citer cet article : M. Boucetta, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Version française abrégée

Le laplacien de Lichnerowicz, introduit dans [3], est un opérateur agissant sur les tenseurs covariants d'une variété riemannienne, il est auto-adjoint, elliptique et respecte la symétrie des tenseurs. Sa restriction aux formes différentielles coincide avec le laplacien de Hodge–de Rham. D'un autre côté, si $\mathcal{G}$ est une algèbre de Lie réelle semisimple compacte, l'élément de Casimir de $\mathcal{G}$ est un élément Ω du centre de l'algèbre enveloppante de $\mathcal{G}$. Il est donné par $\Omega = -\sum_{i=1}^{n} e_i.e_i$ où $(e_1, \ldots, e_n)$ est une base orthonormée de l'opposée de la forme de Killing. Si $\mathcal{G}$ agit sur une variété M, grâce à la dérivée de Lie des champs fondamentaux de cette action, Ω induit un opérateur différentiel Ω_M sur M. Dans cette Note, on s'intéresse aux deux situations suivantes :

(i) Soit G un groupe de Lie compact et semisimple. L'élément de Casimir Ω de l'algèbre de Lie de $\mathcal{G}$ induit sur G deux opérateurs Ω^+ et Ω^- associés, respectivement, à l'action à gauche et à droite de G sur lui même.
(ii) Une paire symétrique est un couple (G, K) où G est un groupe de Lie compact et semisimple et K un sous-groupe fermé de G tels qu'il existe une involution σ de G vérifiant $G_0^\sigma \subset K \subset G^\sigma$, où $G^\sigma = \{g \in G, \sigma(g) = g\}$ et G_0^σ

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est la composante neutre de G^σ. L'action de G sur G/K associe à l'élément de Casimir de l'algèbre de Lie de G un opérateur noté $\Omega_{(G,K)}$.

Le but de cette Note est de démontrer les deux résultats suivants :

Théorème 0.1. *Soit (G, K) une paire symétrique. Alors $\Delta_L = \Omega_{(G,K)}$, où Δ_L est le laplacien de Lichnerowicz sur G/K associé à la métrique riemannienne naturelle sur G/K déduite de la forme de Killing de $\mathcal{G}$.*

Théorème 0.2. *Soit G un groupe de Lie semisimple compact. Alors $\Delta_L = \frac{1}{2}(\Omega^+ + \Omega^-)$, où Δ_L est le laplacien de Lichnerowicz associée à la métrique riemannienne bi-invariante induite par l'opposée de la forme de Killing.*

Le Théorème 0.1 généralise un résultat de [2] et le Théorème 0.2 généralise un résultat énoncé sans preuve dans [1].

1. Introduction and main results

It is well known that the Hodge–de Rham Laplacian on compact symmetric spaces coincides with the Casimir operator (see [2]). Also it is well known that, on a compact semisimple Lie group, the Hodge–de Rham Laplacian is the mean of the left invariant Casimir operator and the right invariant Casimir operator (see [1]). In this paper, we generalize these results to the Lichnerowicz Laplacian. Let us recall the definitions of the Lichnerowicz Laplacian and the Casimir operator and state our main results.

Let (M, g) be a Riemannian n-manifold. For any $p \in \mathbb{N}$, we shall denote by $\Gamma(\bigotimes^p T^*M)$ the space of covariant p-tensor fields on M. The curvature tensor R and the Ricci endomorphism field $r : TM \to TM$ associated to the Levi-Civita connection D are given by

$$R(X,Y)Z = D_{[X,Y]} - (D_X D_Y - D_Y D_X) \quad \text{and} \quad g\bigl(r(X), Y\bigr) = \sum_{i=1}^n g\bigl(R(X, E_i)Y, E_i\bigr),$$

where $(E_1, \dots, E_n)$ is a local orthonormal frame.

The connection D induces a differential operator $D : \Gamma(\bigotimes^p T^*M) \to \Gamma(\bigotimes^{p+1} T^*M)$ given by

$$DT(X, Y_1, \dots, Y_p) = X.T(Y_1, \dots, Y_p) - \sum_{j=1}^p T(Y_1, \dots, D_X Y_j, \dots, Y_p).$$

We denote by $D^* : \Gamma(\bigotimes^{p+1} T^*M) \to \Gamma(\bigotimes^p T^*M)$ its formal adjoint given by

$$D^*T(Y_1, \dots, Y_p) = -\sum_{j=1}^n DT(E_i, E_i, Y_1, \dots, Y_p).$$

The Lichnerowicz Laplacian is the differential operator $\Delta_L : \Gamma(\bigotimes^p T^*M) \to \Gamma(\bigotimes^p T^*M)$ given by

$$\Delta_L(T) = D^*D(T) + R(T),$$

where $R(T)$ is the curvature operator given by

$$\begin{aligned} R(T)(Y_1, \dots, Y_p) = {} & \sum_{j=1}^p T\bigl(Y_1, \dots, r(Y_j), \dots, Y_p\bigr) \\ & - \sum_{l=1}^n \sum_{i<j} \bigl\{ T\bigl(Y_1, \dots, E_l, \dots, R(Y_i, E_l)Y_j, \dots, Y_p\bigr) \\ & + T\bigl(Y_1, \dots, R(Y_j, E_l)Y_i, \dots, E_l, \dots, Y_p\bigr) \bigr\}, \end{aligned}$$

where, in $T(Y_1, \dots, E_l, \dots, R(Y_i, E_l)Y_j, \dots, Y_p)$, E_l takes the place of Y_i and $R(Y_i, E_l)Y_j$ takes the place of Y_j.

This operator, introduced by Lichnerowicz in [3, p. 26], is self-adjoint, elliptic and respects the symmetries of tensor fields. The restriction of Δ_L to the space of differential forms $\Omega^*(M)$ coincides with the Hodge–de Rham Laplacian.

Let $\mathcal{G}$ be real compact semisimple Lie algebra. We denote by $-\langle,\rangle$ the Killing form of $\mathcal{G}$. Recall that the Casimir element of $\mathcal{G}$ is the element Ω belonging to the center of the enveloping algebra $\mathbb{U}(\mathcal{G})$ of $\mathcal{G}$ given by $\Omega = -\sum_{i=1}^n e_i.e_i$, where $(e_1, \dots, e_n)$ is any orthonormal basis with respect to $\langle,\rangle$. If $\mathcal{G}$ acts on a differentiable manifold M, i.e., there exists a Lie algebras homomorphism $\rho : \mathcal{G} \to \mathcal{X}(M)$, Ω gives rise to a differential operator Ω_ρ on M given by, for any $T \in \Gamma(\bigotimes^* T^*M)$, $\Omega_\rho(T) = -\sum_{i=1}^n \mathcal{L}_{\rho(e_i)} \circ \mathcal{L}_{\rho(e_i)} T$, where $\mathcal{L}$ is the Lie derivative. In this Note, we will study the two following cases:

(i) Let G be a compact semisimple Lie group of dimension n and let $\mathcal{G} = T_eG$ be its Lie algebra. For any $u \in \mathcal{G}$ we denote by u^+ and u^- respectively the left and the right invariant vector field associated to u. The Casimir element Ω of $\mathcal{G}$ induces two differential operators on G given by

$$\Omega^+ = -\sum_{i=1}^n \mathcal{L}_{e_i^-} \circ \mathcal{L}_{e_i^-} \quad \text{and} \quad \Omega^- = -\sum_{i=1}^n \mathcal{L}_{e_i^+} \circ \mathcal{L}_{e_i^+},$$

where $(e_1, \dots, e_n)$ is any orthonormal basis with respect to $\langle,\rangle$.

(ii) A compact symmetric pair is a couple (G, K) where G is a compact semisimple Lie group of dimension n and K is a closed subgroup of G such that there exists an involutive automorphism σ such that $G_0^\sigma \subset K \subset G^\sigma$ where $G^\sigma = \{a \in G\colon \sigma(a) = a\}$ and G_0^σ is the connected component of the identity in G^σ. Let $(\mathcal{G}, \mathcal{K})$ be the Lie algebras of (G, K). We have $\mathcal{G} = \mathcal{K} \oplus \mathcal{K}^\perp$ where $\mathcal{K}^\perp$ is the $\langle,\rangle$-orthogonal to $\mathcal{K}$ and

$$[\mathcal{K}, \mathcal{K}^\perp] \subset \mathcal{K}^\perp \quad \text{and} \quad [\mathcal{K}^\perp, \mathcal{K}^\perp] \subset \mathcal{K}. \tag{1}$$

We denote by $\pi : G \to G/K$ the natural projection, we put $o = \pi(e)$ and we consider the natural action of G on G/K. For any vector $u \in \mathcal{G}$, we denote by $\tilde{u}$ the corresponding fundamental vector field on G/K given by $\tilde{u}(\pi(a)) = \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \exp(tu).\pi(a)$. The Casimir element Ω of $\mathcal{G}$ and the action of G on the quotient G/K give rise to a differential operator $\Omega_{(G,K)}$ on G/K given by $\Omega_{(G,K)} = -\sum_{i=1}^n \mathcal{L}_{\tilde{e}_i} \circ \mathcal{L}_{\tilde{e}_i}$, where $(e_1, \dots, e_n)$ is any orthonormal basis of $\langle,\rangle$. On the other hand, the restriction of $\langle,\rangle$ to $\mathcal{K}^\perp$ gives rise to an invariant Riemannian metric g_B on G/K and, for any $u \in \mathcal{G}$, $\tilde{u}$ is a Killing vector field. We can state now our main results.

Theorem 1.1. *Let (G, K) be a compact symmetric pair. Then*

$$\Delta_L = \Omega_{(G,K)},$$

where Δ_L is the Lichnerowicz Laplacian on G/K associated to the Riemannian metric g_B.

A particular case of a compact symmetric pair is given by the Cartesian product $G \times G$ of a compact semisimple Lie group G and its diagonal $\Delta(G) = \{(a, a) \in G \times G\colon a \in G\}$. By applying Theorem 1.1 to this pair, we will deduce the following result:

Theorem 1.2. *Let G be a compact semisimple Lie group. Then*

$$\Delta_L = \frac{1}{2}(\Omega^+ + \Omega^-),$$

where Δ_L is the Lichnerowicz Laplacian on G associated to the biinvariant Riemannian metric on G induced by the negative of the Killing form.

In Section 2, we give a complete proof of Theorems 1.1 and 1.2.

2. Proofs of Theorems 1.1 and 1.2

The proof of Theorem 1.1 requires some preparation. The proof of the following lemma is straightforward:

Lemma 2.1. *Let $(\mathcal{G}, \langle,\rangle)$ be a real Lie algebra endowed with a biinvariant scalar product, i.e., for any $u, v, w \in \mathcal{G}$, $\langle [u, v], w\rangle + \langle v, [u, w]\rangle = 0$. Then*:

(i) *for any bilinear form ω on $\mathcal{G}$, for any $u, v \in \mathcal{G}$ and for any orthonormal basis $(e_1, \dots, e_n)$, we have*

$$\sum_{i=1}^{n} \omega\big([u, e_i], [v, e_i]\big) = \sum_{i=1}^{n} \omega\big(\big[[v, e_i], u\big], e_i\big) = \sum_{i=1}^{n} \omega\big(e_i, \big[[u, e_i], v\big]\big);$$

(ii) *for any subalgebra $\mathcal{K}$ satisfying* (1), *for any orthonormal basis $(e_1, \dots, e_r)$ of $\mathcal{K}$, any orthonormal basis $(f_1, \dots, f_s)$ of $\mathcal{K}^\perp$ and for any $u \in \mathcal{K}^\perp$, we have $\sum_{i=1}^{r}[e_i, [e_i, u]] = \sum_{i=1}^{s}[f_i, [f_i, u]]$.*

Let us give now some properties of the Riemannian manifold $(G/K, g_B)$ where (G, K) is a compact symmetric pair. We denote by D the Levi-Civita connection associated to g_B.

Note first that, for any $u, v \in \mathcal{K}^\perp$ and for any $z \in \mathcal{K}$,

$$g_B\big([\tilde{z}, \tilde{u}], \tilde{v}\big)(o) + g_B\big([\tilde{z}, \tilde{v}], \tilde{u}\big)(o) = 0. \tag{2}$$

On the other hand, from (1), we get that, for any $u, v \in \mathcal{K}^\perp$,

$$[\tilde{u}, \tilde{v}](o) = 0. \tag{3}$$

Moreover, since for any $u, v, w \in \mathcal{G}$, $\tilde{u}$, $\tilde{v}$, $\tilde{w}$ are Killing vector fields we have

$$2g_B(D_{\tilde{u}}\tilde{v}, \tilde{w}) = g_B\big([\tilde{u}, \tilde{v}], \tilde{w}\big) + g_B\big([\tilde{u}, \tilde{w}], \tilde{v}\big) + g_B\big([\tilde{v}, \tilde{w}], \tilde{u}\big). \tag{4}$$

We deduce from (3) and (4) that, for any $u, v \in \mathcal{K}^\perp$,

$$(D_{\tilde{u}}\tilde{v})(o) = 0. \tag{5}$$

The following lemma summarizes some well-known results on curvature of symmetric spaces:

Lemma 2.2. *Let $(G/K, g_B)$ be the Riemannian manifold associated with a compact symmetric pair and D its Levi-Civita connection. Then*:

(i) *for any $u, v, w \in \mathcal{K}^\perp$, $(D_{\tilde{u}}D_{\tilde{v}}\tilde{w})(o) = [\tilde{v}, [\tilde{u}, \tilde{w}]](o)$;*
(ii) *for any $u, v, w \in \mathcal{K}^\perp$, the tensor curvature R and the Ricci endomorphism field r of g_B are given by*

$$R(\tilde{u}, \tilde{v})\tilde{w}(o) = \big[[\tilde{u}, \tilde{v}], \tilde{w}\big](o), \quad r(\tilde{u})(o) = -\sum_{j=1}^{s}\big[[\tilde{u}, \tilde{f}_j], \tilde{f}_j\big](o),$$

where $(f_1, \dots, f_s)$ is any orthonormal basis $\mathcal{K}^\perp$.

Proof of Theorem 1.1. We will show that for any covariant tensor field T, $\Delta_L(T) - \Omega_{G,K}(T)$ vanishes at o and we deduce the vanishing of $\Delta_L(T) - \Omega_{(G,K)}(T)$ on G/K since both Δ_L and $\Omega_{(G,K)}$ are invariant. Choose an orthonormal basis $(e_1, \dots, e_r)$ of $\mathcal{K}$ and an orthonormal basis $(f_1, \dots, f_s)$ of $\mathcal{K}^\perp$. Let T be a covariant p-tensor field. We have

$$\Omega_{(G,K)}(T) = -\sum_{i=1}^{r} \mathcal{L}_{\tilde{e}_i} \circ \mathcal{L}_{\tilde{e}_i}(T) - \sum_{j=1}^{s} \mathcal{L}_{\tilde{f}_j} \circ \mathcal{L}_{\tilde{f}_j}(T).$$

By expanding $L_{\tilde{e}_i} \circ L_{\tilde{e}_i}(T)$ and by using the fact that $\tilde{e}_i(o) = 0$, we get for any $u_1, \dots, u_p \in \mathcal{K}^\perp$

$$\begin{aligned}\mathcal{L}_{\tilde{e}_i} \circ \mathcal{L}_{\tilde{e}_i}(T)(\tilde{u}_1, \dots, \tilde{u}_p)(o) = &-\sum_{j=1} T\big(\tilde{u}_1, \dots, \big[[\tilde{e}_i, \tilde{u}_j], \tilde{e}_i\big], \dots, \tilde{u}_p\big)(o) \\ &+ 2\sum_{l<j} T\big(\tilde{u}_1, \dots, [\tilde{e}_i, \tilde{u}_l], \dots, [\tilde{e}_i, \tilde{u}_j], \dots, \tilde{u}_p\big)(o).\end{aligned} \tag{6}$$

On the other hand, by expanding $\mathcal{L}_{\tilde{f}_j} \circ \mathcal{L}_{\tilde{f}_j}(T)$, we get

$$\begin{aligned}\mathcal{L}_{\tilde{f}_j} \circ \mathcal{L}_{\tilde{f}_j}(T)(\tilde{u}_1, \ldots, \tilde{u}_p) = {} & \tilde{f}_j.\tilde{f}_j.T(\tilde{u}_1, \ldots, \tilde{u}_p) - 2\sum_{i=1}^{p} \tilde{f}_j.T\big(\tilde{u}_1, \ldots, [\tilde{f}_j, \tilde{u}_i], \ldots, \tilde{u}_p\big) \\ & - \sum_{i=1}^{p} T\big(\tilde{u}_1, \ldots, \big[[\tilde{f}_j, \tilde{u}_i], \tilde{f}_j\big], \ldots, \tilde{u}_p\big) \\ & + 2\sum_{l<i} T\big(\tilde{u}_1, \ldots, [\tilde{f}_j, \tilde{u}_l], \ldots, [\tilde{f}_j, \tilde{u}_i], \ldots, \tilde{u}_p\big).\end{aligned}$$

Now by expanding $\mathcal{L}_{f_j} T(\tilde{u}_1, \ldots, [\tilde{f}_j, \tilde{u}_i], \ldots, \tilde{u}_p)$ and by using (3), we get

$$\mathcal{L}_{\tilde{f}_j} \circ \mathcal{L}_{\tilde{f}_j}(T)(\tilde{u}_1, \ldots, \tilde{u}_p)(o) = \tilde{f}_j.\tilde{f}_j.T(\tilde{u}_1, \ldots, \tilde{u}_p)(o) + \sum_{i=1}^{p} T\big(\tilde{u}_1, \ldots, \big[[\tilde{f}_j, \tilde{u}_i], \tilde{f}_j\big], \ldots, \tilde{u}_p\big)(o). \tag{7}$$

On the other hand

$$\begin{aligned}& D^*D(T)(\tilde{u}_1, \ldots, \tilde{u}_p)(o) \\ & \quad = \sum_{i=1}^{s}\Bigg(-\tilde{f}_i.\tilde{f}_i.T(\tilde{u}_1, \ldots, \tilde{u}_p)(o) + 2\sum_{j=1}^{p} \tilde{f}_i.T(\tilde{u}_1, \ldots, D_{\tilde{f}_i}\tilde{u}_j, \ldots, \tilde{u}_p)(o) \\ & \qquad + D_{\tilde{f}_i}\tilde{f}_i.T(\tilde{u}_1, \ldots, \tilde{u}_p)(o) - \sum_{j=1}^{p} T(\tilde{u}_1, \ldots, D_{D_{\tilde{f}_i}\tilde{f}_i}\tilde{u}_j, \ldots, \tilde{u}_p)(o) \\ & \qquad - \sum_{j=1}^{p} T(\tilde{u}_1, \ldots, D_{\tilde{f}_i}D_{\tilde{f}_i}\tilde{u}_j, \ldots, \tilde{u}_p)(o) - 2\sum_{l<j} T(\tilde{u}_1, \ldots, D_{\tilde{f}_i}\tilde{u}_l, \ldots, D_{\tilde{f}_i}\tilde{u}_j, \ldots, \tilde{u}_p)(o)\Bigg).\end{aligned}$$

By using (5), the relation $f_i.T(\tilde{u}_1, \ldots, D_{\tilde{f}_i}\tilde{u}_j, \ldots, \tilde{u}_p)(o) = T(\tilde{u}_1, \ldots, D_{\tilde{f}_i}D_{\tilde{f}_i}\tilde{u}_j, \ldots, \tilde{u}_p)(o)$ and Lemma 2.1, we get

$$\begin{aligned}D^*D(T)(\tilde{u}_1, \ldots, \tilde{u}_p)(o) & = \sum_{i=1}^{s}\Bigg(-\tilde{f}_i.\tilde{f}_i.T(\tilde{u}_1, \ldots, \tilde{u}_p)(o) + \sum_{j=1}^{p} T\big(\tilde{u}_1, \ldots, \big[\tilde{f}_i, [\tilde{f}_i, \tilde{u}_j]\big], \ldots, \tilde{u}_p\big)(o)\Bigg) \\ & \stackrel{(7)}{=} -\sum_{i=1}^{s} \mathcal{L}_{\tilde{f}_i} \circ \mathcal{L}_{\tilde{f}_i}(T)(\tilde{u}_1, \ldots, \tilde{u}_p)(o).\end{aligned} \tag{8}$$

By using Lemma 2.2, we get that the curvature operator is given by

$$\begin{aligned}& R(T)(\tilde{u}_1, \ldots, \tilde{u}_p)(o) \\ & \quad = -\sum_{i=1}^{s}\sum_{j=1}^{p} T\big(\tilde{u}_1, \ldots, \big[[\tilde{u}_j, \tilde{f}_i], \tilde{f}_i\big], \ldots, \tilde{u}_p\big)(o) \\ & \qquad - \sum_{l=1}^{s}\sum_{i<j}\big\{T\big(\tilde{u}_1, \ldots, \tilde{f}_l, \ldots, \big[[\tilde{u}_i, \tilde{f}_l], \tilde{u}_j\big], \ldots, \tilde{u}_p\big)(o) + T\big(\tilde{u}_1, \ldots, \big[[\tilde{u}_j, \tilde{f}_l], \tilde{u}_i\big], \ldots, \tilde{f}_l, \ldots, \tilde{u}_p\big)(o)\big\}.\end{aligned}$$

Now, by using Lemma 2.1, the fact that $\tilde{e}_i(o) = 0$, (3) and (6), one can see easily that

$$R(T)(\tilde{u}_1, \ldots, \tilde{u}_p)(o) = -\sum_{i=1}^{r} \mathcal{L}_{\tilde{e}_i} \circ \mathcal{L}_{\tilde{e}_i}(T)(\tilde{u}_1, \ldots, \tilde{u}_p)(o). \tag{9}$$

From (8) and (9) we deduce that Δ_L and $\Omega_{(G,K)}$ coincides at o and by invariance on all G/K which completes the proof. □

Proof of Theorem 1.2. We define the map $\phi : (G \times G)/\Delta(G) \to G$ by $\phi([g, h]) = gh^{-1}$. This map is an isometry between the symmetric space $(G \times G)/\Delta(G)$ and G endowed with the biinvariant metric induced by the negative of

Killing form. Moreover, ϕ is equivariant with respect to the natural action of $G \times G$ on $(G \times G)/\Delta(G)$ and the action of $G \times G$ on G given by $\rho(g,h)a = gah^{-1}$. According to Theorem 1.1, the Lichnerowicz Laplacian of G coincides with the Casimir operator associated to the action ρ. Let compute this operator. Note that the fundamental vector field on G associated to $(u,v) \in \mathcal{G} \times \mathcal{G}$ is the vector field $u^+ - u^-$ and for any orthonormal basis $(e_1, \ldots, e_n)$ of $\mathcal{G}$,

$$\left(\frac{1}{2}(e_1,e_1), \ldots, \frac{1}{2}(e_n,e_n), \frac{1}{2}(e_1,-e_1), \ldots, \frac{1}{2}(e_n,-e_n)\right)$$

is an orthonormal basis of $\mathcal{G} \times \mathcal{G}$. Hence

$$\Omega_\rho = -\frac{1}{4}\sum_{i=1}^n \mathcal{L}_{e_i^+ - e_i^-} \circ \mathcal{L}_{e_i^+ - e_i^-} - \frac{1}{4}\sum_{i=1}^n \mathcal{L}_{e_i^+ + e_i^-} \circ \mathcal{L}_{e_i^+ + e_i^-} = \frac{1}{2}\left(\Omega^+ + \Omega^-\right)$$

which completes the proof. $\square$

References

[1] B.L. Beers, R.S. Millman, The spectra of the Laplace–Beltrami operator on compact, semisimple Lie groups, Amer. J. Math. 99 (4) (1975) 801–807.

[2] A. Ikeda, Y. Taniguchi, Spectra and eigenforms of the Laplacian on S^n and $P^n(\mathbb{C})$, Osaka J. Math. 15 (3) (1978) 515–546.

[3] A. Lichnerowicz, Propagateurs et commutateurs en relativité générale, Inst. Hautes Etude Sci. Publ. Math. 10 (1961) 5–56.