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Algebraic Geometry

The relations among invariants of points on the projective line

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Abstract

We consider the ring of invariants of n points on the projective line. The space $(\mathbb{P}^1)^n//\mathrm{SL}_2$ is perhaps the first nontrivial example of a GIT quotient. The construction depends on the weighting of the n points. Kempe found generators (in the unit weight case) in 1894. We describe the full ideal of relations for all weightings. In some sense, there is only one equation, which is quadratic except for the classical case of the Segre cubic primal, for $n = 6$ and weight 1^6. The cases of up to 6 points are long known to relate to beautiful familiar geometry. The case of 8 points turns out to be richer still. ***To cite this article: B. Howard et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Les relations entre invariants des points sur la droite projective. Nous considérons l'anneau des invariants de n points ordonnés sur la droite projective. L'espace $(\mathbb{P}^1)^n//\mathrm{SL}_2$ est peut-être le premier exemple intéressant d'un quotient GIT. La construction dépend du choix des poids pour les n points. En 1894, Kempe a introduit un ensemble de générateurs (dans le cas où tous les poids sont égaux à 1). Ici, nous décrivons les relations entre les générateurs pour tous les choix de poids. En un sens il n'y a qu'une relation, qui est quadratique sauf dans le cas classique de la cubique de Segre, lorsque $n = 6$ et que les poids sont 1^6. Pour n inférieur ou égal à 6, la géométrie est classique. Le cas $n = 8$ est plus riche encore et est développé dans cet article. ***Pour citer cet article : B. Howard et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Version française abrégée

Il est bien connu que la géométrie des invariants de n points ordonnés, pour $n \leqslant 6$, est très riche. Nous nous demandons s'il existe une structure similaire pour plus de points. Le cas de 8 points, qui s'avère être plus riche encore, peut être décrit en utilisant des constructions classiques. Dans le cas général de n points, avec des poids arbitraires, nous décrivons les générateurs de l'idéal des relations en termes d'une algèbre graphique. En un certain sens, il y a

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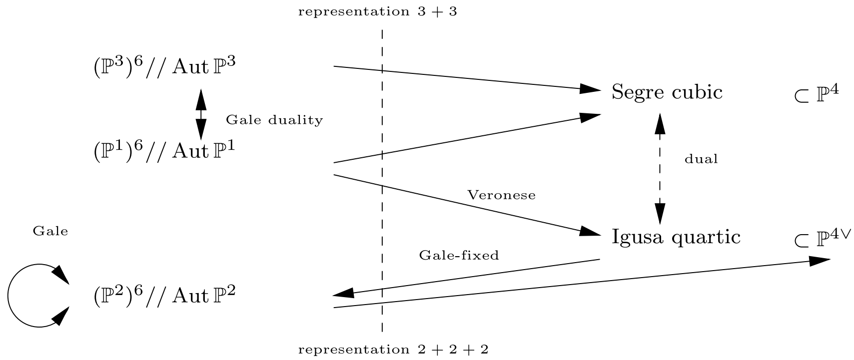

Fig. 1. The classical geometry of six points in projective space.

Fig. 1. La géométrie classique de six points dans l'espace projective.

une seule relation si l'on tient compte des symétries. Quand tous les poids sont égaux à 1 et que n est pair, toutes les relations se déduisent d'une seule par les symétries. En général, la rupture de symétrie donne d'autres équations. La cubique de Segre est trompeuse : dans tous les autres cas, les relations sont quadratiques. La preuve utilise la structure exceptionnelle de $n = 8$ comme point de départ d'une récurrence, puis s'appuie sur la dégénérescence torique de Speyer–Sturmfels et sur la théorie des représentations de S_n.

1. Introduction

We consider the ring of invariants of n points on the projective line, and the GIT quotient $(\mathbb{P}^1)^n//\mathrm{SL}_2$. The quotient depends on a choice of n weights $\mathbf{w} := (w_1, \ldots, w_n) \in (\mathbb{Z}^+)^n$. We assume $2w_i \leqslant \sum w_j$ throughout, to ensure that the GIT quotient is nonempty. The quotient is defined to be $\operatorname{Proj}(\bigoplus R_{kw})$ where $R_{\mathbf{v}} = \Gamma((\mathbb{P}^1)^n, \mathcal{O}(v_1, \ldots, v_n))^{\mathrm{SL}_2}$. Small cases ($n \leqslant 6$) yield familiar beautiful geometry (see [1] for a masterful discussion). In these examples, we take all weights to be 1. Throughout we work over the rationals, though many arguments carry over to an integral setting.

The case $n = 4$ gives the cross ratio $(\mathbb{P}^1)^4 \dashrightarrow \overline{\mathcal{M}}_{0,4} \cong \mathbb{P}^1$. From the perspective of this project, the cross ratio is best understood as a rational map not to $\mathbb{P}^1$, but to the line in $\mathbb{P}^2$ where the three coordinates sum to 0; the S_4-action is more transparent in this way, as we will see shortly.

The case $n = 5$ yields the quintic del Pezzo surface $(\mathbb{P}^1)^5 \dashrightarrow \overline{\mathcal{M}}_{0,5} \hookrightarrow \mathbb{P}^5$. The map is given by the degree 2 invariants; there are no odd degree invariants. The quotient is cut out by 5 quadrics.

The case $n = 6$ is particularly beautiful, see Fig. 1. The ring is generated in degree 1, and this piece has dimension 5, so the quotient threefold, the *Segre cubic*, is naturally a hypersurface in $\mathbb{P}^4$. The equations are cleanest written in six variables $x_1, \ldots, x_6$ as $\sum x_i = \sum x_i^3 = 0$. (This description was given by Joubert in 1867, and in 1911 Coble gave an invariant-theoretical interpretation of Joubert's identities.)

The quotient $(\mathbb{P}^3)^6 // \operatorname{Aut}(\mathbb{P}^3)$ is canonically isomorphic to $(\mathbb{P}^1)^6 // \operatorname{Aut}(\mathbb{P}^1)$, via the Gale transform. (Roughly: interpret the first as a linear map $f : \mathbb{Q}^6 \to \mathbb{Q}^4$, modulo GL_4; then the second corresponds to the dual of $\ker f \to \mathbb{Q}^6$.) The quotient $(\mathbb{P}^2)^6 // \operatorname{Aut}(\mathbb{P}^2)$ is a double cover of $\mathbb{P}^4$, branched over the Igusa quartic hypersurface, given by the equations

$$w_1 + \cdots + w_6 = 4\left(w_1^4 + \cdots + w_6^4\right) - \left(w_1^2 + \cdots + w_6^2\right)^2 = 0.$$

Gale duality exchanges the two sheets of the double cover, and thus the Gale-fixed points, where the six points lie on a conic, correspond to the Igusa quartic. The Igusa quartic is dual (in the sense of classical projective geometry) to the Segre cubic, and the map on moduli sends six points on $\mathbb{P}^1$ to six points on the conic via the second Veronese embedding of $\mathbb{P}^1$.

Our motivating question is the following: *is there similarly rich structure in the case of more points*? In Section 2, we describe a structure for eight points parallel to, and in some sense generalizing, that of six points. Although the results are geometric, the proofs are essentially representation theoretic. In Section 3, we describe the generators for the ideal of relations for n points in general. They are "inherited" from the $n = 8$ case. This completes the program initiated in [3]. Proofs will appear in [4].

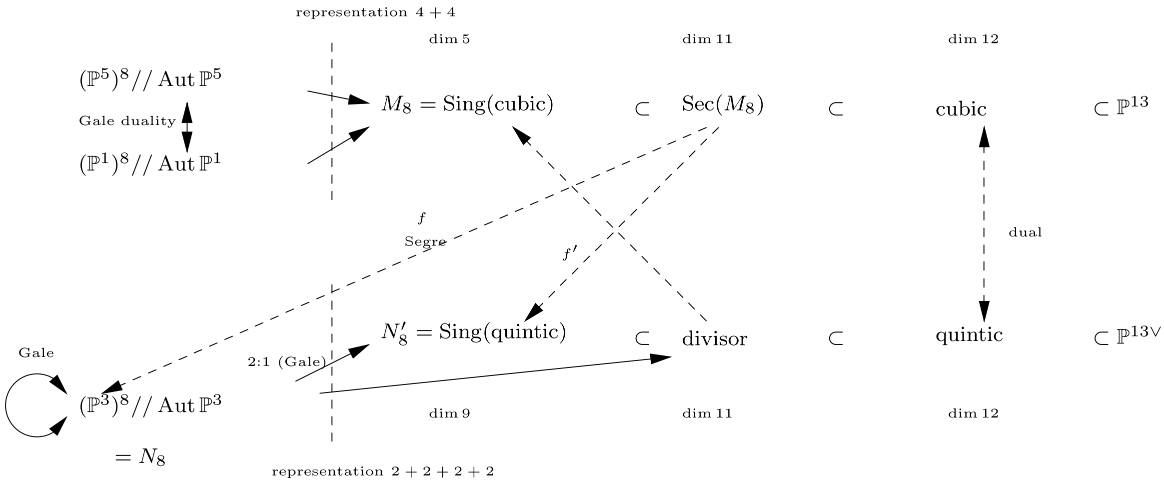

Fig. 2. Relations among moduli spaces of eight points in projective space.

Fig. 2. Relations entre espaces de huit points dans espace projective.

2. Eight points

The quotient $M_8 := (\mathbb{P}^1)^8 // \operatorname{Aut}(\mathbb{P}^1)$ naturally lies in $\mathbb{P}^{13}$. (The graded ring is generated in degree 1, and its degree 1 piece has rank 14.) A number of authors have shown by computer calculation that the ideal of relations is generated by 14 quadrics (Koike, Kondo, Freitag, Salvati Manni, Maclagan, ...). We describe the structure of the quadrics more directly, motivated by a suggestion of Dolgachev. See Fig. 2.

The symmetric group acts on the graded ring. There is a unique skew-invariant cubic, in $\operatorname{Sym}^3 H^0((\mathbb{P}^1)^8, \mathcal{O}(1, \dots, 1))$, and it lies in the ideal of M_8, because it vanishes on the union of the diagonals $x_a = x_b$, of degree 7. (Sam Grushevsky and Riccardo Salvati Manni have pointed out to us that the fact that it contains M_8 readily follows in the language of theta functions, see [2, §5].) More is true: M_8 is the singular locus of the skew cubic; in fact, the affine cone of M_8 is the singular locus of the affine cone of the cubic. Therefore the homogeneous ideal of M_8 is generated by the 14 partial derivatives of the skew cubic. We emphasize that our proof requires no computer verification.

The dual to the skew cubic has surprisingly low degree — it is a skew quintic in $\mathbb{P}^{13}$, whose singular locus N'_8 has dimension 9. The moduli space N_8 of 8 points in $\mathbb{P}^3$ is a double cover of this singular locus. The sheets of the double cover are exchanged by Gale (self-)duality.

By Bezout's theorem, the secant variety to M_8 is contained in the skew cubic; it is a divisor. The duality birational map f from the cubic to the quintic blows down this divisor: given a secant line to M_8 meeting M_8 at points p and q, the 14 quadrics (which give the duality map) vanish at p and q, and thus are scalar multiples of each other. Hence we have identified one of the two dimensions of $\operatorname{Sec} M_8$ contracted by f. We now describe the other dimension contracted, by lifting $f : \operatorname{Sec}(M_8) \dashrightarrow N'_8$ to $\operatorname{Sec}(M_8) \dashrightarrow N_8$. Fix two distinct points of M_8, and hence a secant line. From this data of a pair of octuples of points on $\mathbb{P}^1$, we obtain an octuple of points on $\mathbb{P}^1 \times \mathbb{P}^1$, which via the Segre embedding yields 8 points in $\mathbb{P}^3$ (up to projective equivalence). Conversely, given 8 general points in $\mathbb{P}^3$, we can find a smooth quadric through them (indeed, a one-parameter of smooth quadrics), yielding an unordered pair of octuples of points on $\mathbb{P}^1$. (In fact this construction is used to show that the skew quintic is dual to the skew cubic.)

Even more structure may be present. The trisecant variety to N'_8 is contained in the skew quintic, and the quadrisecant variety to N'_8 is contained in the divisor. A dimension estimate shows that this is quite unusual; both should easily fill out the full $\mathbb{P}^{13\vee}$. One might expect that both containments are equality, although we have not yet shown this.

This generalizes the $n = 6$ story in a number of ways. For example, the analogue of the skew cubic for $n = 8$ is the Segre cubic for $n = 6$. (The analogue for $n = 4$ is the union of the three boundary points.) Also, the similarity between Figs. 1 and 2 is not coincidental: one can describe a fibration over the constructions of Fig. 1 in the boundary of Fig. 2, commuting with all dualities. For example, if we consider octuples of points in $\mathbb{P}^3$ where the two points coincide, projecting from those two points yields six points in $\mathbb{P}^2$, and the Gale dualities in the two figures correspond.

3. The general case

We describe the ring of invariants by means of graphs. To a directed graph Γ (with no loops, but multiple edges allowed) on n ordered vertices (in bijection with the n points), we associate

$$\prod_{\mathbf{ab}\in\Gamma}(x_a y_b - y_a x_b),$$

an invariant element of $\mathcal{O}(\mathbf{v})$, where $\mathbf{v}$ is the n-tuple of valences of the vertices. The degree $\mathbf{w}$ invariants are generated (as a vector space) by these elements. This description can be used to show that the ring of invariants is generated in lowest possible degree. In the unit weight case, this is Kempe's Theorem [5].

Remark. Weyl's theorems on rings of invariants in [7] are of the form: *First Main Theorem:* describe generators of the ring of invariants; *Second Main Theorem:* describe relations among the generators. One of his main results is for the symplectic group acting diagonally on the direct sum of n copies of the standard representation ([7, Thm. 6.1.A and 6.1.B] are the first and second main theorems respectively); for the case of SL_2, we obtain the affine cone over the Grassmannian $\mathrm{Gr}(2, n)$. The first main theorem gave generators for the invariants, the Plücker coordinates. The second main theorem gave the relations, the Plücker relations. Kempe [5] proved the first main theorem when the direct sum is replaced by the tensor product. Theorem 3.1 (or more correctly the main theorem of [4]) is the second main theorem.

We make a series of observations about this graphical algebra:

Multiplication. Multiplication of graphs is by superposition. (See for example Fig. 3(a). The vertex labels 1 through 4 are omitted for simplicity. In later figures, even the vertices will be left implicit.)

Sign relations. Changing the orientation of an edge changes the sign of the invariant (e.g. Fig. 3(b)).

Plücker relation. Direct calculation shows the relation of Fig. 3(c).

Bigger relations from smaller ones. The "four-point" Plücker relation immediately "extends" to relations among more points, e.g. Fig. 3(d) for 6 points. Any relation may be extended in this way. For example, the sign relation in general should be seen as an extension of the two-point sign relation.

Remark. The sign and (extended) Plücker relations generate all the linear relations. They can be used to show that the "non-crossing graphs" (with no pairs of edges crossing) with all edges oriented "upwards" (the arrow points toward the higher-numbered label) form a basis — the graphical form of the straightening algorithm. But breaking symmetry obscures the structure of the ring.

The Segre cubic. The relation of Fig. 3(e) is patently true: the superposition of the three graphs on the left is the same as that of the three graphs on the right. This is a cubic relation on the six-point space. It turns out to be nonzero, and is thus necessarily the Segre cubic relation. Of course, all that matters about the orientations of the edges is that they are the same on the both sides of the equation.

The skew cubic on eight points. One can describe the Segre cubic differently, as $\sum_{\sigma\in S_6}\mathrm{sgn}(\sigma)\sigma(\Gamma^3)$ where Γ is any 1-regular graph on 6 vertices. Replacing 6 by 8 yields the skew cubic of Section 2.

A simple (binomial) quadric on eight points. Fig. 3(f) gives an obvious relation on 8 points. The arrowheads are omitted for simplicity; they should be chosen consistently on both sides, as in Fig. 3(e).

Simple quadrics for at least eight points are obtained by "extending" the eight-point relations, e.g. Fig. 3(g) is the extension to 12 points, where the same two edges are added to each graph in Fig. 3(f).

We may now state the main theorem of [4], in a special case:

Theorem 3.1 *(Main theorem for the n even "unit weight" case* $\mathbf{w} = 1^n$*). If* $n \neq 6$*, the simple quadrics* (*i.e. the* S_n*-orbit of the quadric above*) *generate the ideal of relations.*

By [3, Thm. 1.2], the arbitrary weight case readily reduces to the "unit weight" case $\mathbf{w} = 1^n$ (n even), so this solves the problem for arbitrary weight. For example, an explicit description of the quadrics in the del Pezzo case of five points are as the five rotations of the patently true relation in Fig. 3(h).

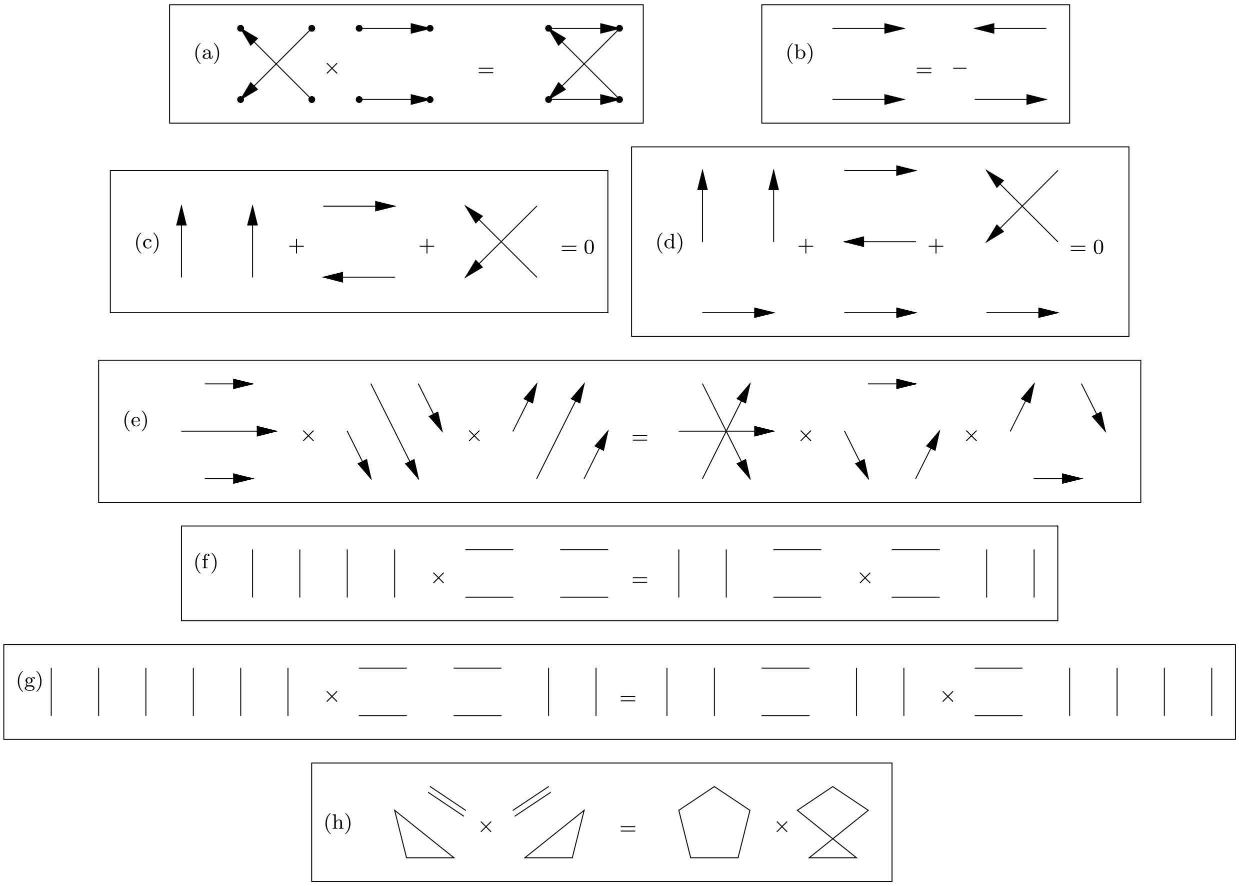

Fig. 3. Relations in the graphical algebra.

Fig. 3. Relations dans l'algèbre graphique.

Overview of proof. We use a construction of Speyer–Sturmfels [6] to degenerate the quotient into a toric variety, and we show the toric variety is cut out in degrees two and three. We identify the cubics, and lift them to explicit cubic relations for the original (quotient) variety. We show by representation theory (and elementary combinatorics) that they lie in the ideal cut out by the quadrics — this is the most difficult part of the proof and uses the $n = 8$ case as the base case of an inductive argument. Then we use representation theory to see that the quadrics are generated as a module by the simple quadrics defined above.

Conclusion. We have thus answered our motivating question: there is sufficient structure in the general case that we can describe (generators of) the relations completely. The structure is inherited from the case of $n = 8$, where it is a consequence of exceptional geometry.

Future prospects. Many phenomena observed in our study of n points on $\mathbb{P}^1$ are present when $\mathbb{P}^1$ is replaced by an arbitrary variety. The third author has developed a formalism to describe the situation and has formulated conjectures about the coordinate rings of $X^n /\!/ G$ as n varies.

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References

[1] I. Dolgachev, D. Ortland, Point sets in projective space and theta functions, Astérisque 165 (1988), 210 pp. (1989).
[2] E. Freitag, R. Salvati Manni, The modular variety of hyperelliptic curves of genus three, 2007, preprint, arXiv:0710.5920v1.
[3] B. Howard, J. Millson, A. Snowden, R. Vakil, The equations for the moduli space of n points on the line, Duke Math. J. 146 (2) (2009) 175–226.
[4] B. Howard, J. Millson, A. Snowden, R. Vakil, The ideal of relations for the ring of invariants of n points on the line, in preparation.
[5] A. Kempe, On regular difference terms, Proc. London Math. Soc. 25 (1894) 343–350.
[6] D. Speyer, B. Sturmfels, The tropical Grassmannian, Adv. Geom. 4 (3) (2004) 389–411.
[7] H. Weyl, The Classical Groups: Their Invariants and Representations, Princeton U.P., Princeton, NJ, 1997.