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Partial Differential Equations

Schrödinger equations with indefinite weights in the whole space

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Abstract

We consider in this Note equations defined in $\mathbb{R}^N$ involving Schrödinger operators with indefinite weight functions and with potentials which tend to infinity at infinity. We give some results for the existence of principal eigenvalues and for the maximum principle. We also obtain Courant–Fischer formulas for such eigenvalues. ***To cite this article: L. Cardoulis, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Équations de Schrödinger à poids indéfinis définies dans tout l'espace. On considère dans cette Note des équations définies sur $\mathbb{R}^N$ avec des opérateurs de Schrödinger à poids indéfinis dont les potentiels tendent vers l'infini à l'infini. On donne des résultats pour l'existence de valeurs propres principales ainsi que pour le principe du maximum. On obtient aussi des formules de type Courant–Fischer pour ces valeurs propres. ***Pour citer cet article : L. Cardoulis, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Version française abrégée

Considérons l'équation de Schrödinger (1). L'objet de cette Note est d'étudier l'existence, éventuellement l'unicité, de valeurs propres principales quand la fonction poids m est indéfinie et quand le potentiel q tend vers l'infini à l'infini. On donne aussi une formule de type Courant–Fischer pour ces valeurs propres principales. On considèrera deux types d'hypothèses pour le poids m, soit $m \in L^\infty(\mathbb{R}^N)$ soit $m \in L^{\frac{N}{2}}(\mathbb{R}^N) \cap L^\infty_{loc}(\mathbb{R}^N)$ et $N \geqslant 3$. On commencera par étudier le cas d'un poids positif. Dans le cas d'un poids positif borné, on obtient l'existence et l'unicité d'une valeur propre principale positive en utilisant une version générale du théorème de Krein–Rutman (cf. [5]) qui nous permet de travailler dans $L^2(\mathbb{R}^N)$ dont le cône positif est d'intérieur vide. Pour un poids positif dans $L^{\frac{N}{2}}(\mathbb{R}^N)$ $(N \geqslant 3)$, l'idée pour obtenir l'existence d'une valeur propre principale positive est de définir la forme bilinéaire $\beta(u, v) = \int_{\mathbb{R}^N} muv$ puis d'appliquer le théorème de Riesz. On étend ensuite ces résultats au cas d'un poids m qui change de signe dans $\mathbb{R}^N$ en réécrivant (1) en une forme équivalente avec une fonction poids positive et un paramètre dépendant de λ. Pour cela on utilise les formules de Courant–Fischer des valeurs propres principales obtenues dans le cas de poids positifs. On

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obtient ainsi l'existence et l'unicité d'une valeur propre principale positive ainsi que d'une valeur propre principale négative, ainsi que leurs caractérisations variationnelles. Enfin on donne des résultats pour le principe du maximum dans le cas de l'équation linéaire (2). Tous ces résultats étendent des résultats bien connus pour l'opérateur Laplacien $-\Delta$, bien sûr dans un domaine borné (cf. [6,7]), mais aussi dans $\mathbb{R}^N$ (cf. [2,3]). Notons que la même approche peut être utilisée pour obtenir l'existence d'une valeur propre principale globale pour un système défini dans $\mathbb{R}^N$ avec des opérateurs de Schrödinger à poids indéfinis (cela sera étudié dans une autre note). Notons enfin que certaines extensions des principes du maximum et de l'antimaximum ont été obtenues pour l'opérateur $-\Delta+q$ dans $\mathbb{R}^N$ sans fonction poids (cf. [1]), résultats qui peuvent être généralisés dans le cas de poids positif ; mais à notre connaissance le principe de l'antimaximum généralisé énoncé dans [1] n'a pas encore été étendu au cas de poids indéfinis.

1. Introduction

We consider in this Note the Schrödinger operator $-\Delta+q$ defined in $\mathbb{R}^N$ where q is a potential which satisfies the following hypothesis:

($\mathbf{H_q^1}$) $q \in L^2_{loc}(\mathbb{R}^N) \cap L^p_{loc}(\mathbb{R}^N)$, $p > \frac{N}{2}$, such that $\lim_{|x|\to+\infty} q(x) = +\infty$ and $q \geqslant \text{cst} > 0$.

Mainly, this paper deals with the existence of principal eigenvalues for the problem

$$(-\Delta+q)u = \lambda m u \quad \text{in } \mathbb{R}^N, \tag{1}$$

where λ is a real parameter and m denotes an indefinite weight function.

The variational space, denoted by $V_q(\mathbb{R}^N)$, is the completion of $D(\mathbb{R}^N)$, the set of C^∞ functions with compact supports, with respect to the norm

$$\|u\|_q = \left(\int_{\mathbb{R}^N} \left[|\nabla u|^2 + q u^2\right]\right)^{1/2}.$$

We recall that the embedding of $V_q(\mathbb{R}^N)$ into $L^2(\mathbb{R}^N)$ is compact.

We will consider two different types of hypotheses for the weight m, either $m \in L^\infty(\mathbb{R}^N)$ or $m \in L^{\frac{N}{2}}(\mathbb{R}^N) \cap L^\infty_{loc}(\mathbb{R}^N)$, $N \geqslant 3$. For a positive weight m, we obtain the existence, uniqueness and a Courant–Fischer formula for a positive principal eigenvalue $\lambda_{1,q,m}$. For a weight m which changes sign in $\mathbb{R}^N$, we obtain the existence, uniqueness and a Courant–Fischer formula for a positive principal eigenvalue $\lambda_{1,q,m}$ and also for a negative principal eigenvalue $\tilde{\lambda}_{1,q,m}$. This extends well-known results for the Laplacian operator $-\Delta$, of course in a bounded domain (see [6,7]) but also in $\mathbb{R}^N$ (see [2,3]). We conclude by giving the maximum principle for the equation

$$(-\Delta+q)u = \lambda m u + f \quad \text{in } \mathbb{R}^N, \tag{2}$$

where $f \in L^2(\mathbb{R}^N)$. Note that extensions of maximum and antimaximum principles, respectively called ground state positivity and negativity, are given in [1] for the Schrödinger operator $-\Delta+q$ on $\mathbb{R}^N$ without any weight m; these results can be generalized for the Schrödinger operator $-\Delta+q$ on $\mathbb{R}^N$ with a positive bounded weight function but to our knowledge, the antimaximum principle for the Schrödinger operator $-\Delta+q$ associated with an indefinite weight m in the whole space, has not been achieved yet. See also [7] for the antimaximum principle for the Laplacian $-\Delta$ on a bounded domain, in the case of an indefinite weight.

Our Note is organized as follows. In Section 2 we recall and complete some previous results for the Schrödinger operator $-\Delta+q$ associated with a positive weight m in the whole space (see [4]; existence of principal eigenvalues and maximum principle). For a positive bounded weight m, as in [6] we obtain the existence and the uniqueness of a positive principal eigenvalue by using a general version of the Krein–Rutman Theorem given in [5] which allows us to work in $L^2(\mathbb{R}^N)$ whose cone has empty interior. For a positive weight $m \in L^{\frac{N}{2}}(\mathbb{R}^N) \cap L^\infty_{loc}(\mathbb{R}^N)$, as in [2,3], the idea to get the existence of a principal eigenvalue, is to define the bilinear form $\beta(u,v) = \int_{\mathbb{R}^N} muv$ for all u, v in $V_q(\mathbb{R}^N)$, then to apply the Riesz Theorem. Section 3 is devoted to the study of the Schrödinger operator $-\Delta+q$ associated with a weight m which changes sign in $\mathbb{R}^N$: we still give results for the existence of principal eigenvalues and for the maximum principle. We adapt here the ideas expressed in [6] to our case by rewriting (1) in an equivalent form with a positive weight and a parameter function depending upon λ. Then the proof is reduced to a fixed point problem which needs the variational characterization of the corresponding principal eigenvalue obtained in the case of a positive weight. Finally, note that the same approach can be used to get the existence of a global principal eigenvalue for a system involving Schrödinger operators in $\mathbb{R}^N$ (this will be studied in another note), already there is a great number of results for the maximum principle for a system.

2. The case of a positive weight

First, we assume that the weight m satisfies the following hypothesis:

($\mathbf{H_m^1}$) There exist two positive reals α and β such that $0 < \alpha \leqslant m \leqslant \beta$ in $\mathbb{R}^N$.

Thus, if m satisfies ($\mathbf{H_m^1}$), $\|\cdot\|_m$ (defined by $\|u\|_m^2 = \int_{\mathbb{R}^N} mu^2$ for all $u \in L^2(\mathbb{R}^N)$) is a norm in $L^2(\mathbb{R}^N)$, equivalent to the usual norm. We let M the multiplication operator given by the function m. Since the operator $(-\Delta + q)^{-1}M : (L^2(\mathbb{R}^N), \|\cdot\|_m) \to (L^2(\mathbb{R}^N), \|\cdot\|_m)$ is positive self-adjoint and compact, then its spectrum is discrete and consists of a positive sequence tending to 0. We denote by $\lambda_{1,q,m}$ the inverse of the first (largest) eigenvalue of the operator $(-\Delta + q)^{-1}M$. So we have [4, Proposition 1.1]

Theorem 2.1. *Assume that m satisfies* ($\mathbf{H_m^1}$). *Then the eigenvalue* $\lambda_{1,q,m}$ *is simple and there exists an associated function* $\phi_{1,q,m}$ *which is a strictly positive and continuous function in* $\mathbb{R}^N$. *And we have*

$$
\begin{aligned}
&(-\Delta + q)\phi_{1,q,m} = \lambda_{1,q,m} m\phi_{1,q,m} \quad \text{in } \mathbb{R}^N, \quad \lambda_{1,q,m} > 0, \quad \phi_{1,q,m} > 0; \\
&\lambda_{1,q,m} = \inf\left\{ \frac{\int_{\mathbb{R}^N}[|\nabla\phi|^2 + q\phi^2]}{\int_{\mathbb{R}^N} m\phi^2},\ \phi \in D\big(\mathbb{R}^N\big) \right\}.
\end{aligned} \tag{3}
$$

If now we consider a weaker hypothesis for the weight m

($\mathbf{H_m^{*1}}$) $0 < m \leqslant \text{cst}$ in $\mathbb{R}^N$.

Then the operator $(-\Delta + q)^{-1}M : (L^2(\mathbb{R}^N), \|\cdot\|_{L^2(\mathbb{R}^N)}) \to (L^2(\mathbb{R}^N), \|\cdot\|_{L^2(\mathbb{R}^N)})$ is compact and, due to the strong maximum principle for the operator $-\Delta + q$ in $\mathbb{R}^N$, is also strongly positive in the sense of quasi-interior points in $L^2(\mathbb{R}^N)$ in the sense of Daners and Koch–Medina (cf. [5]; this allows us to work in $L^2(\mathbb{R}^N)$ whose cone has empty interior). This implies that $(-\Delta+q)^{-1}M$ is irreductible and we can apply the Krein–Rutman Theorem, more precisely Theorem 12.3, page 118 from [5]. We deduce that $r((-\Delta+q)^{-1}M)$, the spectral radius of $(-\Delta+q)^{-1}M$, is a strictly positive and simple eigenvalue associated with an eigenfunction ψ which is a quasi-interior point of $L^2(\mathbb{R}^N)$, that is $\psi > 0$ in $\mathbb{R}^N$. Of course $\lambda_{1,q,m} = \frac{1}{r((-\Delta+q)^{-1}M)}$, $\phi_{1,q,m} = \psi$ and $r((-\Delta + q)^{-1}M)$ is the only one eigenvalue of $(-\Delta + q)^{-1}M$ associated with a positive eigenfunction. So Theorem 2.1 is also available when m satisfies ($\mathbf{H_m^{*1}}$).

For a locally bounded weight which satisfies the following hypothesis:

($\mathbf{H_m^2}$) $m \in L^{N/2}(\mathbb{R}^N) \cap L^\infty_{loc}(\mathbb{R}^N)$ $(N \geqslant 3)$, $meas\{x \in \mathbb{R}^N,\ m(x) > 0\} \neq 0$.

We have the following result:

Theorem 2.2. *Assume that m satisfies* ($\mathbf{H_m^2}$). *Then there exists a unique principal eigenvalue* $\lambda_{1,q,m}$ *associated with a positive eigenfunction* $\phi_{1,q,m}$ *which satisfy* (3).

Proof. We proceed as in [2] or [3]. We denote by $\langle u, v\rangle_q = \int_{\mathbb{R}^N}[\nabla u.\nabla v + quv]$ the inner product in $V_q(\mathbb{R}^N)$. We set the bilinear form

$$\beta(u, v) = \int_{\mathbb{R}^N} muv \quad \text{for all } u, v \in V_q\big(\mathbb{R}^N\big).$$

Since $N \geqslant 3$, the embedding of $H^1(\mathbb{R}^N)$ into $L^{2^*}(\mathbb{R}^N)$ is continuous where $2^* = \frac{2N}{N-2}$ and so the embedding of $V_q(\mathbb{R}^N)$ into $L^{2^*}(\mathbb{R}^N)$ is continuous too. Therefore β is a bilinear continuous form. From the Riesz Theorem, we get the existence of a continuous operator $L : V_q(\mathbb{R}^N) \to V_q(\mathbb{R}^N)$ which satisfies

$$\beta(u, v) = \langle Lu, v\rangle_q \quad \text{for all } u, v \in V_q\big(\mathbb{R}^N\big).$$

As in [2,3], we can prove that the self-adjoint operator L is compact. So the largest eigenvalue of L is given by

$$\mu_1 = \sup_{u \in V_q(\mathbb{R}^N); u \neq 0} \frac{\langle Lu, u\rangle_q}{\langle u, u\rangle_q} = \sup_{u \in V_q(\mathbb{R}^N);\, u \neq 0} \frac{\int_{\mathbb{R}^N} mu^2}{\int_{\mathbb{R}^N} [|\nabla u|^2 + qu^2]}.$$

Since $m > 0$ in a subset Ω of $\mathbb{R}^N$ with nonzero measure, by choosing a function in $V_q(\mathbb{R}^N)$ with support included in Ω, we get that $\mu_1 > 0$ and so $\lambda_{1,q,m} = \frac{1}{\mu_1} > 0$. We denote by ϕ_1 an eigenfunction associated with μ_1 for the operator L. Clearly ϕ_1 is still an eigenfunction associated with $\lambda_{1,q,m}$ for the operator $-\Delta + q$ associated with m. Note also that ϕ_1 is a continuous function in $\mathbb{R}^N$. Moreover we can prove that there exists an eigenfunction associated with μ_1 which does not change sign; we still denote it by ϕ_1 and we have $\phi_1 \geqslant 0$. By the Harnack Inequality for the operator $-\Delta + q - \lambda_{1,q,m} m$, we get that $\phi_1 > 0$. Besides, we can prove that μ_1 is a simple eigenvalue using a method developed for example in [4, Proposition 1.1] and we easily obtain the following variational characterization for $\lambda_{1,q,m}$

$$\lambda_{1,q,m} = \frac{1}{\mu_1} = \inf\left\{ \frac{\int_{\mathbb{R}^N} [|\nabla\phi|^2 + q\phi^2]}{\int_{\mathbb{R}^N} m\phi^2},\ \phi \in V_q\big(\mathbb{R}^N\big) \text{ s.t.} \int_{\mathbb{R}^N} m\phi^2 > 0 \right\}. \tag{4}$$

Finally, from $m \geqslant 0$ in $\mathbb{R}^N$ we prove the uniqueness of such an eigenvalue, i.e. of a positive principal eigenvalue and this concludes the proof. □

Now we recall the classical weak maximum principle

Theorem 2.3. *Assume that m satisfies* ($\mathbf{H^1_m}$) *or* ($\mathbf{H^{*1}_m}$) *or* ($\mathbf{H^2_m}$), $f \geqslant 0$ *and u is a solution of Eq.* (2). *If* $\lambda < \lambda_{1,q,m}$, *then* $u \geqslant 0$.

3. The case of an indefinite weight

We consider here, as in Section 2, different hypotheses for the weight m.

($\mathbf{H'^1_m}$) $m \in L^\infty(\mathbb{R}^N)$, m is positive in an open subset $\Omega_m^+ = \{x \in \mathbb{R}^N, m(x) > 0\}$ with nonzero measure and m is negative in an open subset $\Omega_m^- = \{x \in \mathbb{R}^N, m(x) < 0\}$ with nonzero measure.

($\mathbf{H'^2_m}$) $m \in L^{N/2}(\mathbb{R}^N) \cap L^\infty_{loc}(\mathbb{R}^N)$ ($N \geqslant 3$), $meas(\Omega_m^+) > 0$, $meas(\Omega_m^-) > 0$.

Theorem 3.1. *Assume that m satisfies* ($\mathbf{H'^1_m}$) *or* ($\mathbf{H'^2_m}$). *Then the operator* $-\Delta + q$ *associated with the weight m has a unique positive principal eigenvalue* $\lambda_{1,q,m}$ *associated with a positive eigenfunction* $\phi_{1,q,m}$ *and* $(\lambda_{1,q,m}, \phi_{1,q,m})$ *satisfy* (3) *and* (4). *Moreover this operator has a unique negative principal eigenvalue* $\tilde{\lambda}_{1,q,m}$ *associated with a positive eigenfunction* $\tilde{\phi}_{1,q,m}$ *and we have*

$$(-\Delta + q)\tilde{\phi}_{1,q,m} = \tilde{\lambda}_{1,q,m} m \tilde{\phi}_{1,q,m} \quad \text{in } \mathbb{R}^N, \quad \tilde{\lambda}_{1,q,m} = -\lambda_{1,q,-m} < 0, \quad \tilde{\phi}_{1,q,m} > 0;$$
$$\tilde{\lambda}_{1,q,m} = \sup\left\{ \frac{\int_{\mathbb{R}^N} [|\nabla\phi|^2 + q\phi^2]}{\int_{\mathbb{R}^N} m\phi^2},\ \phi \in V_q\big(\mathbb{R}^N\big) \text{ s.t.} \int_{\mathbb{R}^N} m\phi^2 < 0 \right\}.$$

Proof. As in Theorem 2.2, we can prove the existence of $\lambda_{1,q,m}$ a principal positive eigenvalue (since m is positive in Ω_m^+) and of $\tilde{\lambda}_{1,q,m}$ a principal negative eigenvalue (since m is negative in Ω_m^-) by changing m by $-m$ since $(-\lambda)(-m) = \lambda m$ for either the case of ($\mathbf{H'^1_m}$) or ($\mathbf{H'^2_m}$). But this method does not allow us to conclude for the uniqueness of such eigenvalues because of the last step in Theorem 2.2 which necessities a weight which does not change sign.

So we adapt to our case a method of [6] to obtain existence and uniqueness of a principal positive eigenvalue and of a principal negative eigenvalue.

(i) Assume here that m satisfies ($\mathbf{H'^1_m}$). We denote by $\Omega_m^0 = \{x \in \mathbb{R}^N, m(x) = 0\}$. Let u be a solution of (1). For given $\lambda > 0$, we rewrite (1) as an eigenvalue problem with parameter $\sigma(\lambda)$

$$(-\Delta + Q_\lambda)u = \sigma(\lambda)\rho u \quad \text{in } \mathbb{R}^N, \tag{5}$$

where 1_Ω denotes the characteristic function of $\Omega := \Omega_m^0 \cup \Omega_m^-$, $Q_\lambda := q + \lambda(m^- + 1_\Omega)$ and $\rho := m^+ + 1_\Omega$. From Theorem 2.1, we deduce that Eq. (5) admits a unique principal eigenvalue $\sigma(\lambda)$ which is strictly positive and associated with a principal eigenfunction $\phi_{1,\lambda} > 0$. Note that $\lambda > 0$ is an eigenvalue of (1) if and only if $\sigma(\lambda) = \lambda$ and from the variational characterization (3), we have:

$$\sigma(\lambda) = \inf_{\phi \in V_q(\mathbb{R}^N)} \left\{ \frac{\int_{\mathbb{R}^N} [|\nabla\phi|^2 + q\phi^2] + \lambda \int_{\mathbb{R}^N} (m^- + 1_\Omega)\phi^2}{\int_{\mathbb{R}^N} (m^+ + 1_\Omega)\phi^2} \right\}.$$

So $\sigma(\lambda) = \lambda_{1,Q_\lambda,\rho} < \lambda^+_{1,Q_\lambda,\rho}$ where $\lambda_{1,Q_\lambda,\rho}$ is the principal eigenvalue of the operator $-\Delta + Q_\lambda$ associated with the weight ρ in $\mathbb{R}^N$ and $\lambda^+_{1,Q_\lambda,\rho}$ is the principal eigenvalue of the operator $-\Delta + Q_\lambda$ associated with the weight ρ in Ω_m^+ with Dirichlet boundary condition. Noticing that $\lambda^+_{1,Q_\lambda,\rho} = \lambda^+_{1,q,m^+}$ we get that $\sigma(\lambda)$ is bounded respectively to λ. Note also that $\sigma(\lambda)$ is increasing and continuous with respect to λ and that $\sigma(0) = \lambda_{1,q,\rho} > 0$ since $\rho > 0$. Therefore from the continuity of the bounded and increasing function σ and from $\sigma(0) > 0$, we deduce that $\sigma(\lambda)$ intersects at least the diagonal and so there exists $\tilde\lambda > 0$ such that $\sigma(\tilde\lambda) = \tilde\lambda$. As in [6] we can prove that $\tilde\lambda$ is unique. We obtain the same results for $\lambda < 0$ by changing m^+ with m^- since $\lambda m = (-\lambda)(-m)$.

(ii) Assume now that m satisfies ($\mathbf{H'^2_m}$). Let a function $h > 0$, h fixed which satisfies ($\mathbf{H^2_m}$). For $\lambda > 0$, we study the following equation with parameter $r(\lambda)$:

$$\left(-\Delta + q + h + \lambda m^-\right)u = r(\lambda)\left(m^+ + \frac{1}{\lambda}h\right)u \quad \text{in } \mathbb{R}^N. \tag{6}$$

Note that $\lambda \neq 0$ is an eigenvalue of (1) if and only if $r(\lambda) = \lambda$. We now denote by $\tilde Q_\lambda := q + h + \lambda m^-$ and $\tilde\rho_\lambda := m^+ + \frac{1}{\lambda}h$. Note that $\tilde Q_\lambda$ satisfies ($\mathbf{H^1_q}$) and $\tilde\rho_\lambda$ satisfies ($\mathbf{H^2_m}$), $\tilde\rho_\lambda > 0$. From Theorem 2.2, we deduce the existence of a principal eigenvalue $r(\lambda) > 0$ for (6) which is simple and associated with a positive eigenfunction and $r(\lambda)$ is the only eigenvalue which is positive and principal for the operator $-\Delta + \tilde Q_\lambda$ associated with the weight $\tilde\rho_\lambda$ in $\mathbb{R}^N$. Moreover from (4) we have:

$$r(\lambda) = \inf_{\phi \in V_q(\mathbb{R}^N); \phi \neq 0} \left\{ \frac{\int_{\mathbb{R}^N} [|\nabla\phi|^2 + (q+h)\phi^2 + \lambda m^- \phi^2]}{\int_{\mathbb{R}^N} (m^+ + \frac{1}{\lambda}h)\phi^2} \right\}.$$

As in [6], we study the properties of the function r (note that r is continuous bounded function, $r(0) = \lim_{\lambda \to 0^+} r(\lambda) = 0$). Note also that $r'(0) > 1$. Indeed,

$$r'(0) = \lim_{\lambda \to 0^+} \frac{r(\lambda)}{\lambda} = \lim_{\lambda \to 0^+} \inf_{\phi \in V_q(\mathbb{R}^N); \phi \neq 0} \left\{ \frac{\int_{\mathbb{R}^N} [|\nabla\phi|^2 + (q+h)\phi^2 + \lambda m^- \phi^2]}{\int_{\mathbb{R}^N} (\lambda m^+ + h)\phi^2} \right\}$$

and $r'(0) = \lambda_{1,q+h,h}$ is the principal eigenvalue of the operator $-\Delta + q + h$ (associated with the weight h) in $\mathbb{R}^N$. Since $\lambda_{1,q+h,h} = \lambda_{1,q,h} + 1$, we obtain that $r'(0) > 1$. So $r(\lambda)$ intersects at least the diagonal. Thus there exists $\lambda^* > 0$ such that $r(\lambda^*) = \lambda^*$ and the uniqueness of such λ^* follows from the characterization of $r(\lambda^*)$. □

Finally, we give the maximum principle for (2).

Theorem 3.2. *Assume that m satisfies ($\mathbf{H'^1_m}$) or ($\mathbf{H'^2_m}$), $\lambda \in \mathbb{R}$, $f \in L^2(\mathbb{R}^N)$, $f \geqslant 0$ and u is a solution of Eq. (2). If $\tilde\lambda_{1,q,m} < \lambda < \lambda_{1,q,m}$, then $u \geqslant 0$.*

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