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Families of special Weierstrass points

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Abstract

The purpose of this Note is to show that loci of (special) Weierstrass points on the fibers of a family $\pi:\mathfrak{X}\to S$ of smooth curves of genus $g\geqslant 2$ can be studied by simply pulling back the Schubert calculus naturally living on a suitable Grassmann bundle over $\mathfrak{X}$. Using such an idea we prove new results regarding the decomposition in $A_*(\mathfrak{X})$ of the class of the locus of Weierstrass points having weight at least 3 as the sum of classes of Weierstrass points having "bounded from below" gaps sequences. ***To cite this article: L. Gatto, P. Salehyan, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Familles de points de Weierstrass speciaux. L'objectif principal de cette Note est de montrer que les lieux de points de Weierstrass speciaux dans une famille générale de courbes lisses $\mathfrak{X}\to S$ de genre $g\geqslant 2$ peuvent être étudiés simplement en tirant en arrière le calcul de Schubert qui vit naturellement dans une fibrée opportune de Grassmann. En utilisant cette idée nous obtenons des nouveaux résultats concernant la décomposition de la classe dans $A_*(\mathfrak{X})$ du lieu des points de Weierstrass qui ont poids au moins 3 comme somme des classes de points de Weierstrass avec suites particulières de lacunes. ***Pour citer cet article : L. Gatto, P. Salehyan, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

1. Introduction

Let $\pi:\mathfrak{X}\to S$ be a family of smooth complex projective curves of genus $g\geqslant 2$, parametrized by some regular scheme S and let K_π be the relative canonical sheaf of π. For each $h\geqslant -1$, we denote by J^hK_π the vector bundle of rank $h+1$ over $\mathfrak{X}$ of the hth relative jets of K_π: we set $J^0K_\pi=K_\pi$ and $J^{-1}K_\pi=0$. Let $\mathbb{E}_\pi:=\pi_*\omega_\pi$ be the Hodge bundle of the family, a locally free sheaf of rank g over S and, for each $i\geqslant 0$, let $\lambda_i:=c_i(\mathbb{E}_\pi)\in A^i(S)$ be the ith Chern class of the bundle $\mathbb{E}_\pi$. For each $h\geqslant -1$ we denote by $\partial_{h,P}$ the natural evaluation map $(\pi^*\mathbb{E}_\pi)_P\to J^h_PK_\pi$, sending $(P,\omega)\in(\pi^*\mathbb{E}_\pi)_s$ to $\partial_{h,P}\omega$, which locally on the fiber over s is the evaluation of $\omega\in H^0(\mathfrak{X}_s,K_{\pi(P)})$ at P together with its first h derivatives (here $\pi(P)=s$). Let $\partial_h:\pi^*\mathbb{E}_\pi\to J^hK_\pi$ be the bundle map gotten by patching together all such evaluation maps. Clearly the map $\partial_{2g-2}:\pi^*\mathbb{E}_\pi\to J^{2g-2}K_\pi$ is a vector bundle monomorphism, because if

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$\omega \in H^0(\mathfrak{X}_{\pi(P)}, K_{\pi(P)})$, $\partial_{2g-2}(P,\omega) = \partial_{2g-2,P}\omega = 0$ if and only if ω vanishes at P with multiplicity at least $2g-1$, hence ω must be zero for degree reasons. For each $k \geqslant 1$, let $\rho_k : G(k, J^{2g-2}K_\pi) \to \mathfrak{X}$ denote the Grassmann bundle of k-planes in the fibers of ρ_k and let $0 \to \mathcal{S}_k \to \rho_k^* J^{2g-2}K_\pi \to \mathcal{Q}_k \to 0$ be the universal exact sequence over it: $\mathcal{S}_k$ is the rank k universal subbundle of $\rho_k^* J^{2g-2}K_\pi$ and $\mathcal{Q}_k$ is the universal quotient of it (of rank $2g-1-k$). Thus the monomorphism ∂_{2g-2} induces a section $\iota_g : \mathfrak{X} \to G(g, J^{2g-2}K_\pi)$ of ρ_g defined as:

$$\iota_g(P) = \partial_{2g-2,P}\big(H^0(\mathfrak{X}_s, K_{\mathfrak{X}_s})\big) \subseteq G\big(g, J_P^{2g-2}K_\pi\big), \tag{1}$$

where, for each $s \in S$ and each $P \in \pi^{-1}(s)$, $\partial_{2g-2,P}(H^0(\mathfrak{X}_s, K_{\mathfrak{X}_s}))$ denotes the $\partial_{2g-2,P}$-monomorphic image of $H^0(\mathfrak{X}_{\pi(P)}, K_{\mathfrak{X}_s})$ inside the vector space $J_P^{2g-2}K_\pi$. By the universal property of Grassmann schemes, it turns out that $\iota_g^*\mathcal{S}_g$ is isomorphic to π^*E_π, while $\iota_g^*(\rho_g^* J^{2g-2}K_\pi) = J^{2g-2}K_\pi$, i.e. the map ∂_{2g-2} is the pull-back of the universal map $0 \to \mathcal{S}_g \xrightarrow{\tau_g} \rho_g^* J^{2g-2}K_\pi$. For each $j \geqslant i$, let $p_{j,i}$ be the natural projection $J^{j-1}K_\pi \to J^{i-1}K_\pi$ and let $p_i := p_{2g-1,i}$. For each $k \geqslant 1$, denote by $\boldsymbol{\varepsilon} : \mathcal{S}_k \to \rho_k^* K_\pi$ the map $p_1 \circ \tau_k$; furthermore, let $\partial_j \boldsymbol{\varepsilon} := p_{j+1} \circ \tau_k$, $1 \leqslant j \leqslant h+1$. The composition $\partial_h \boldsymbol{\varepsilon} : \mathcal{S}_k \to \rho_k^* J^{2g-2}K_\pi$, of p_{2g-1} with τ_k, will be identified with the monomorphism τ_k *tout court*. For each $k \geqslant 1$ define $\partial_{-1}\boldsymbol{\varepsilon} = 0$ and, for each $1 \leqslant i_1 < \cdots < i_k \leqslant 2g-1$, let $\Omega_{(i_1,\ldots,i_k)}(J^\bullet K_\pi)$ be the set of all $\Lambda \in G(k, J^{2g-2}K_\pi)$ such that $\mathrm{rk}_\Lambda(\partial_{i_j-2}\boldsymbol{\varepsilon}) \leqslant j-1$, for all $j \in \{1,\ldots,k\}$. In particular, $\Omega_{(i_1)}(J^\bullet K_\pi) = \{\mathbf{u} \in \mathcal{S}_1 \mid \mathrm{rk}_{\mathbf{u}}(\partial_{i_1-2}\boldsymbol{\varepsilon}) = 0\}$.

A local analysis shows that the expected codimension of the Schubert variety $\Omega_{(i_1,\ldots,i_k)}(J^\bullet K)$ of $G(k, J^{2g-2}K_\pi)$ is $(i_1-1)+\cdots+(i_k-k)$. In particular, $\Omega_{(i)}(J^\bullet K)$, $1 \leqslant i \leqslant 2g-1$ has the expected codimension $i-1$. It is the zero locus of the vector bundle map $\partial_{i-2}\boldsymbol{\varepsilon} : \mathcal{S}_1 \to \rho_1^* J^{i-2}K_\pi$. Thinking of $\partial_{i-2}\boldsymbol{\varepsilon}$ as a section of the bundle $\rho_1^* J^{i-2}K_\pi \otimes \mathcal{S}_1^\vee$, define

$$\epsilon^i := \big[\Omega_i\big(J^\bullet K\big)\big] \cap \big[\mathbb{P}\big(J^{2g-2}K_\pi\big)\big] = c_{i-1}\big(\rho_1^* J^{i-2}K_\pi \otimes \mathcal{S}_1^\vee\big) \in A_*\big(\mathbb{P}\big(J^{2g-2}K_\pi\big)\big).$$

Notice that ϵ^1 is the fundamental class of $A_*(\mathbb{P}(J^{2g-2}K_\pi))$. It is easy to show that $(\epsilon^1,\ldots,\epsilon^{2g-2},\epsilon^{2g-1})$ is an $A^*(\mathfrak{X})$-basis of $A_*(\mathbb{P}(J^{2g-2}K_\pi))$. In fact, for each $1 \leqslant i \leqslant 2g-1$, $\epsilon^i = \mu^i + \sum_{j=1}^{i-1} \rho_1^* c_j(J^{i-2}K_\pi)\mu^{i-j}$, where $\mu^i := c_1(\mathcal{S}_1)^i \cap [\mathbb{P}(J^{2g-2}K_\pi)]$, and $(\mu^1,\ldots,\mu^{2g-1})$ is a basis of $A_*(\mathbb{P}(J^{2g-2}K_\pi))$ [1, Example 8.3.4].

By virtue of [7] (see also [3], [4]) $\{\epsilon^{i_1} \wedge \cdots \wedge \epsilon^{i_k} \mid 1 \leqslant i_1 < \cdots < i_k \leqslant 2g-1\}$ is a basis of $A_*(G(k, J^{2g-2}K_\pi))$ as well.

Theorem 1.1. *For each $k \geqslant 1$ and each $1 \leqslant i_1 < \cdots < i_k \leqslant 2g-1$ one has:*

$$\big[\Omega_{(i_1,\ldots,i_k)}\big(J^\bullet K_\pi\big)\big] = \epsilon^{i_1} \wedge \cdots \wedge \epsilon^{i_k} \in \bigwedge^k A_*\big(\mathbb{P}\big(J^{2g-2}K_\pi\big)\big) \cong A_*\big(G\big(k, J^{2g-2}K_\pi\big)\big).$$

Proof. We are applying [10, Theorem 7.3] by Laksov and Thorup rephrased by considering flags of quotient bundles instead of subbundles. See also [5] (and [9] for general background). □

2. Application to loci of Weierstrass points

Let $WGS(P) = \langle 1, n_2, \ldots, n_g\rangle$ be the Weierstrass gap sequence at $P \in \mathfrak{X}_{\pi(P)}$. We say that $WGS(P) \succcurlyeq (i_1,\ldots,i_g)$ if and only if $n_j \geqslant i_j$ for each $1 \leqslant j \leqslant g$. It is easy to check that $\iota_g(P) \in \Omega_{(i_1,\ldots,i_g)}(J^\bullet K_\pi)$ if and only if $WGS(P) \succcurlyeq (i_1,\ldots,i_g)$. Denote by $V_\pi(i_1,\ldots,i_g) = \{P \in \mathfrak{X} \mid WGS(P) = (i_1,\ldots,i_g)\}$ and with $W_\pi(i_1,\ldots,i_g) = \{P \in \mathfrak{X} \mid WGS(P) \succcurlyeq (i_1,\ldots,i_g)\}$. Clearly $V_\pi(i_1,\ldots,i_g) \subseteq W_\pi(i_1,\ldots,i_g)$. We give $W_\pi(i_1,\ldots,i_g)$ the scheme structure induced by the Schubert varieties $\Omega_{(i_1,\ldots,i_g)}(J^\bullet K_\pi)$, i.e. $W_\pi(i_1,\ldots,i_g) = \iota_g^{-1}(\Omega_{(i_1,\ldots,i_g)}(J^\bullet K)) \cong \Omega_{(i_1,\ldots,i_g)}(J^\bullet K_\pi) \cap \iota_g(\mathfrak{X})$.

As a consequence of the definition of $W_\pi(I)$ and of Theorem 1.1, one sees that if $W_\pi(i_1,\ldots,i_g)$ has the expected codimension $\sum_{j=1}^g (i_j - j)$, then $[W_\pi(i_1,\ldots,i_g)] = \iota_g^*(\epsilon^{i_1} \wedge \cdots \wedge \epsilon^{i_g})$. Notice that if $P \in \mathfrak{X}$ is a hyperelliptic Weierstrass point then it obviously belongs to $W_\pi(1,3,4,\ldots,g+1)$. Conversely, if $P \in W_\pi(1,3,4,\ldots,g+1)$, then $WGS(P) \succcurlyeq (1,3,4,\ldots,g+1)$. Since, the complement of $WGS(P)$ must be a semigroup, the only admissible Weierstrass gap sequence greater than $(1,3,4,\ldots,g+1)$ is $(1,3,5,\ldots,2g-1)$. In this case one has that $W_\pi(1,3,4,\ldots,g+1) = V_\pi(1,3,\ldots,2g-1)$ and then:

$$\begin{aligned}\big[V_\pi(1,3,\ldots,2g-1)\big] &= \big[W_\pi(1,3,4,\ldots,g+1)\big] \\ &= \iota_g^*\big[\Omega_{(1,3,4,\ldots,g+1)}(J^\bullet K_\pi)\big] = \iota_g^*\big(\epsilon^1 \wedge \epsilon^3 \wedge \cdots \wedge \epsilon^{2g-1}\big).\end{aligned} \tag{2}$$

Expressing the right-hand side of (2) in terms of the basis $(\mu^1, \ldots, \mu^g)$ and pulling back via ι_g, one gets precisely the expression displayed in [13, p. 314]. Computational details will appear in [6]. Furthermore, if $\pi : \mathfrak{X} \to S$ is the "universal curve" $\mathcal{C}_g \to M_g$, then

$$\mathcal{C}_g = W_\pi(1, \ldots, g) \supseteq W_\pi(1, 2, \ldots, g-1, g+1) \supseteq W_\pi(1, 2, \ldots, g-2, g+1) \supseteq \cdots \supseteq W_\pi(1, 3, 4, \ldots, g+1)$$

and

$$\begin{aligned} M_g &= \pi\big(W_\pi(1, \ldots, g)\big) \supseteq \pi\big(W_\pi(1, 2, \ldots, g-1, g+1)\big) \\ &\supseteq \pi\big(W_\pi(1, 2, \ldots, g-2, g+1)\big) \supseteq \cdots \supseteq \pi\big(W_\pi(1, 3, 4, \ldots, g+1)\big) \end{aligned}$$

are precisely the Arbarello's flag as in [13, p. 310].

3. The classes of $W_\pi(I)$

We present now the main results of our forthcoming [6] (cf. [11,12]). Let $\pi : \mathfrak{X} \to S$ be a family of smooth curves of genus $g \geqslant 4$ parametrized by a smooth scheme S of dimension at least 2. By [8, Theorem 3.7], each irreducible component of the locus $Vwt(3)(\pi) := \{P \in \mathfrak{X} \mid P \text{ is a Weierstrass point of weight} \geqslant 3\}$ has the expected codimension 3. This locus is not irreducible. Indeed, at least set theoretically:

$$Vwt(3) = W_\pi(I_{g,1}) \cup W_\pi(I_{g,2}) \cup W_\pi(I_{g,3})$$

where, for notational simplicity, we set:

$$I_{g,1} = (1, \ldots, g-3, g-1, g, g+1), \quad I_{g,2} = (1, 2, \ldots, g-2, g, g+2), \quad \text{and}$$
$$I_{g,3} = (1, \ldots, g-1, g+3).$$

Here is a higher codimensional analogous of [8, Theorem 4.6] (see also [2]):

Theorem 3.1. *The equality* $[Vwt(3)] = [W_\pi(I_{g,1})] + 2[W_\pi(I_{g,2})] + [W_\pi(I_{g,3})]$ *holds in* $A_*(\mathfrak{X})$.

Sketch of a proof. The class of $Vwt(3)$ was computed in [8, Proposition 4.9] (is the expression into the brackets at the end of the proof). The expression of the three summands occurring into the decomposition of $[Vwt(3)]$ are also easily computable by using standard Schubert calculus on Grassmann bundles, possibly in the form [7], and pulling back via i_g. Then one checks that the left-hand side coincides with the sum occurring on the right-hand side (see the example in genus 4 below). □

The piece of Schubert calculus necessary to predict and prove the decomposition stated in Theorem 3.1 and many other similar decompositions will be explained in [6]. By the way, all of them can be checked by direct computations. Notice that one can find expressions of new classes (in $A_*(\mathfrak{X})$) already in genus 4. In this case one has the special case of Theorem 3.1 for $g = 4$:

$$\big[Vwt(3)(\pi)\big] = \iota_4^*\big(\epsilon^1 \wedge \epsilon^3 \wedge \epsilon^4 \wedge \epsilon^5\big) + 2\iota_4^*\big(\epsilon^1 \wedge \epsilon^2 \wedge \epsilon^4 \wedge \epsilon^6\big) + \iota_4^*\big(\epsilon^1 \wedge \epsilon^2 \wedge \epsilon^3 \wedge \epsilon^7\big), \tag{3}$$

and it may be easily checked that:

$$\begin{aligned} \iota_4^*\big(\epsilon^1 \wedge \epsilon^3 \wedge \epsilon^4 \wedge \epsilon^5\big) &= 15K^3 - 7K^2\pi^*\lambda_1 + 3K\pi^*\lambda_2 - \pi^*\lambda_3, \\ \iota_4^*\big(\epsilon^1 \wedge \epsilon^2 \wedge \epsilon^4 \wedge \epsilon^6\big) &= 285K^3 - 90K^2\pi^*\lambda_1 + \big(9\pi^*\lambda_2 - 6(\pi^*\lambda_1)^2\big)K + \pi^*\lambda_1\pi^*\lambda_2 - \pi^*\lambda_3, \qquad (4) \\ \iota_4^*\big(\epsilon^1 \wedge \epsilon^2 \wedge \epsilon^3 \wedge \epsilon^7\big) &= 735K^3 - 175K^2\pi^*\lambda_1 + 21K\big((\pi^*\lambda_1)^2 - \pi^*\lambda_2\big) \\ &\quad + (\pi^*\lambda_1)^3 - 2\pi^*\lambda_1\pi^*\lambda_2 + \pi^*\lambda_3, \qquad (5) \end{aligned}$$

where, for notational brevity, we have set $K := \rho_4^* c_1(K_\pi)$. Notice that (3) is precisely the expression for the class of hyperelliptic points in fibers of π which occurs into braces at [13, p. 278], fourth line from the top, computed for $g = 4$. Expression (5) can be computed applying Porteous' formula to the natural map $\pi^*\mathbb{E}_\pi \to J^{g+1}K_\pi$ (see [8]) and the class of $W_\pi(1, 2, 4, 6)$ (i.e. the class of $W(I_{g,2})$ for $g = 4$) is new. Furthermore substituting such values into (3) one gets $[Vwt(3)(\pi)] = 1320K^3 - 362K^2\pi^*\lambda_1$, the known expression computed in [8] (formula at the bottom of p. 2252 for $g = 4$).

The decomposition stated in Theorem 3.1, whose (3) is the particular case for $g = 4$, is new as well. Obviously $V_\pi(1,3,5,7)$ is closed in $\mathfrak{X}$, because $W_\pi(1,3,4,5) = V_\pi(1,3,5,7)$ and $W_\pi(1,3,5,7)$ is closed. In [6] we shall prove that $V_\pi(1,2,4,7)$, $V_\pi(1,2,3,7)$ are closed as well, being (possibly union of) irreducible components of $W_\pi(1,2,4,6)$ and $W_\pi(1,2,3,7)$.

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