



Group Theory

Algebras of invariant differential operators on a class of multiplicity free spaces

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Received 10 March 2009; accepted after revision 14 October 2009

Available online 30 October 2009

Presented by Michel Duflo

Abstract

Let G be a connected reductive algebraic group and let $G' = [G, G]$ be its derived subgroup. Let (G, V) be a multiplicity free representation with a one-dimensional quotient (see definition below). We prove that the algebra $D(V)^{G'}$ of G' -invariant differential operators with polynomial coefficients on V , is a quotient of a so-called Smith algebra over its center. Over \mathbb{C} this class of algebras was introduced by S.P. Smith (1990) as a class of algebras similar to $\mathcal{U}(\mathfrak{sl}_2)$. Our result generalizes the case of the Weil representation, where the associative algebra generated by $Q(x)$ and $Q(\partial)$ (Q being a non-degenerate quadratic form on V) is a quotient of $\mathcal{U}(\mathfrak{sl}_2)$. Other structure results are obtained when (G, V) is a regular prehomogeneous vector space of commutative parabolic type. **To cite this article:** *H. Rubenthaler, C. R. Acad. Sci. Paris, Ser. I 347 (2009)*.

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Résumé

Algèbres d'opérateurs différentiels invariants associés à certains espaces sans multiplicités. Soit G un groupe algébrique réductif connexe et soit $G' = [G, G]$ son groupe dérivé. Soit (G, V) un espace sans multiplicités ayant un quotient unidimensionnel (voir la définition ci-dessous). Nous montrons que l'algèbre $D(V)^{G'}$ des opérateurs différentiels à coefficients polynomiaux G' -invariants sur V , est isomorphe à un quotient d'une algèbre de Smith sur son centre. Sur \mathbb{C} cette classe d'algèbres, avait été introduite par S.P. Smith (1990) comme une classe d'algèbres semblables à $\mathcal{U}(\mathfrak{sl}_2)$. Notre résultat généralise le cas de la représentation de Weil, où l'algèbre associative engendrée par $Q(x)$ et $Q(\partial)$ (Q étant une forme quadratique non dégénérée sur V), est un quotient de $\mathcal{U}(\mathfrak{sl}_2)$. D'autres résultats de structure sont obtenus lorsque (G, V) est un espace préhomogène parabolique commutatif régulier. **Pour citer cet article :** *H. Rubenthaler, C. R. Acad. Sci. Paris, Ser. I 347 (2009)*.

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1. Introduction

For any smooth affine algebraic variety M , we shall denote by $\mathbb{C}[M]$ the algebra of regular functions on M , and by $\mathbf{D}(M)$ the algebra of differential operators on M . Moreover, if an algebraic group H acts on M , we will denote respectively by $\mathbb{C}[M]^H$ and $\mathbf{D}(M)^H$ the algebras of H -invariant regular functions and of H -invariant differential operators on M .

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Let $Q(x)$ be a non-degenerate quadratic form on \mathbb{C}^n and let $Q(\partial)$ be the differential operator with constant coefficients defined by the condition $(Q(\partial)Q)(0) = 1$. It is well known that these two differential operators generate a Lie subalgebra of $\mathbf{D}(\mathbb{C}^n)$ which is isomorphic to \mathfrak{sl}_2 (note that this Lie algebra completely determines the infinitesimal Weil representation of $\mathfrak{sp}(n, \mathbb{C})$). Consequently the associative subalgebra $\mathbb{C}[Q(x), Q(\partial)]$ of $\mathbf{D}(\mathbb{C}^n)$, generated by these two operators, is a quotient of $\mathcal{U}(\mathfrak{sl}_2)$. A natural question one can ask is the following: what happens if $Q(x)$ is replaced by an arbitrary homogeneous polynomial P of degree n ? J.I. Igusa [3] has proved that if $\partial^\circ P \geq 3$, then the Lie algebra generated by $P(x)$ and $P(\partial)$ is infinite-dimensional. In this paper we basically study the case where $P(x) = \Delta_0(x)$ is the fundamental relative invariant of a so-called multiplicity-free space (G, V) with one-dimensional quotient ([6], see 2.1 below). Among several results concerning various algebras of differential operators, our main result is Theorem 2.8 below which says that the algebra $\mathbf{D}(V)^{G'}$ is a quotient of a so-called Smith algebra over its center. As Smith algebras are “similar” to $\mathcal{U}(\mathfrak{sl}_2)$, this can be viewed as a generalization of the infinitesimal Weil representation. The structure of the algebra of radial components of $\mathbf{D}(V)^{G'}$ is obtained as a corollary.

2. Invariant differential operators on multiplicity free spaces with one-dimensional quotient

Let G be a reductive algebraic group over \mathbb{C} , and let $G' = [G, G]$ be its derived group. Let (G, V) be a rational finite-dimensional linear representation of G . Recall that (G, V) is said to be *multiplicity free* if the associated representation of G on $\mathbb{C}[V]$ decomposes without multiplicities. For the classification and properties of multiplicity free spaces we refer to [1,5,4]. Let us first recall that a multiplicity free space is necessarily a prehomogeneous vector space (abbreviated *PV*) not only under G but also under any Borel subgroup B of G .

Definition 2.1. (See Levasseur [6].) A multiplicity-free space (G, V) is said to have a one-dimensional quotient if there exists a polynomial Δ_0 such that $\Delta_0 \notin \mathbb{C}[V]^G$ and such that $\mathbb{C}[V]^{G'} = \mathbb{C}[\Delta_0]$.

From now on we assume that (G, V) is a multiplicity free space with a one-dimensional quotient. We remark first that the polynomial Δ_0 is necessarily irreducible. There exists then a non-trivial character χ_0 of G , such that $\forall g \in G, \forall x \in V, \Delta_0(gx) = \chi_0(g)\Delta_0(x)$. In other words Δ_0 is a relative invariant of the *PV* (G, V) , with character χ_0 . We define $\Omega = \{x \in V \mid \Delta_0(x) \neq 0\}$. The algebra of regular functions on Ω is given by $\mathbb{C}[\Omega] = \{f = \frac{P}{\Delta_0^k} \mid P \in \mathbb{C}[V], k \in \mathbb{N}\}$. From the definitions it is easy to see that the natural action of G on $\mathbb{C}[\Omega]$ is still without multiplicities. As an easy consequence of this fact, one can prove the following result:

Proposition 2.2. *One has $\mathbf{D}(\Omega)^G = \mathbf{D}(V)^G$ (that is, any G -invariant differential operator with regular coefficients on Ω has polynomial coefficients). Moreover this algebra is commutative.*

Hence we have the following inclusions among the algebras $\mathbf{D}(V)^G = \mathbf{D}(\Omega)^G, \mathbf{D}(V)^{G'}, \mathbf{D}(\Omega)^{G'}$:

$$\begin{array}{ccc} \mathbf{D}(V)^G & \xlongequal{\quad} & \mathbf{D}(\Omega)^G \\ \downarrow & & \downarrow \\ \mathbf{D}(V)^{G'} & \longrightarrow & \mathbf{D}(\Omega)^{G'} \end{array}$$

In the case where (G, V) is a regular *PV* of commutative parabolic type, or equivalently when V is a simple Jordan algebra over \mathbb{C} and G its structure group, the result of Proposition 2.2 was essentially known by computing explicit generators of $\mathbf{D}(\Omega)^G$ [7,10].

Let us define: $\mathcal{T}_0 = \mathbf{D}(V)^G = \mathbf{D}(\Omega)^G, \mathcal{T} = \mathbf{D}(\Omega)^{G'}$.

Let us also set:

$$X = \Delta_0(x), \quad X^{-1} = \frac{1}{\Delta_0(x)}, \quad Y = \Delta_0(\partial), \quad E = \text{Euler operator}$$

where $\Delta_0(\partial)$ is the differential operator with constant coefficients on V defined by the condition $(\Delta_0(\partial)\Delta_0)(0) = 1$. Clearly $E \in \mathcal{T}_0 = \mathbf{D}(V)^G, X, Y \in \mathbf{D}(V)^{G'},$ and $X^{-1} \in \mathbf{D}(\Omega)^{G'} = \mathcal{T}$.

Definition 2.3. The automorphism τ of the algebra \mathcal{T} is defined by $\forall D \in \mathcal{T}, \tau(D) = XDX^{-1}$.

It is easy to see that \mathcal{T}_0 is stable under τ . The preceding definition can also be read $XD = \tau(D)X$ for $D \in \mathcal{T}$. In fact a similar relation is true for Y : for all $D \in \mathcal{T}$, we have $DY = Y\tau(D)$.

Let us denote by $\mathcal{T}_0[X, Y]$ the subalgebra of \mathcal{T} generated by \mathcal{T}_0 , X , and Y and by $\mathcal{T}_0[X, X^{-1}]$ the subalgebra of \mathcal{T} generated by \mathcal{T}_0 , X , and X^{-1} .

Proposition 2.4. *Let d_0 be the degree of Δ_0 . We have:*

- 1) $\mathbf{D}(V)^{G'} = \mathcal{T}_0[X, Y] = \bigoplus_{p \in \mathbb{N}} \mathcal{T}_0 X^p \oplus \bigoplus_{k \in \mathbb{N}^*} \mathcal{T}_0 Y^k$, and moreover $\mathcal{T}_0 X^p = \mathcal{T}_0[X, Y]_p = \{D \in \mathcal{T}_0[X, Y] \mid [E, D] = pd_0 D\}$, $p \in \mathbb{N}$, $\mathcal{T}_0 Y^k = \mathcal{T}_0[X, Y]_k = \{D \in \mathcal{T}_0[X, Y] \mid [E, D] = -kd_0 D\}$, $k \in \mathbb{N}^*$.
- 2) $\mathcal{T} = \mathbf{D}(\Omega)^{G'} = \mathcal{T}_0[X, X^{-1}] = \bigoplus_{p \in \mathbb{Z}} \mathcal{T}_0 X^p$, and moreover $\mathcal{T}_0 X^p = \mathcal{T}_p = \{D \in \mathcal{T} \mid [E, D] = pd_0 D\}$, $p \in \mathbb{Z}$.

Remark 2.5. Assertion 1) is due to Levasseur [6, Th. 4.11]. In the particular case of PV 's of commutative parabolic type the result was obtained independently by the author [8].

Let now $\mathcal{Z}(\mathcal{T})$ be the center of \mathcal{T} . It can be shown that $\mathcal{Z}(\mathcal{T})$ is also the center of $\mathbf{D}(V)^{G'} = \mathcal{T}_0[X, Y]$ and that $D \in \mathcal{Z}(\mathcal{T}) \Leftrightarrow D \in \mathcal{T}_0$ and $\tau(D) = D$.

The key result is then the following:

Theorem 2.6.

- 1) $\mathcal{T}_0 = \mathcal{Z}(\mathcal{T}) \oplus E\mathcal{T}_0$.
- 2) Any element $H \in \mathcal{T}_0$ can be uniquely written in the form $H = H_0 + EH_1 + \dots + E^k H_k$, $H_i \in \mathcal{Z}(\mathcal{T})$, $k \in \mathbb{N}$.

Sketch of proof. Let B be a Borel subgroup of G . As mentioned before the space (B, V) is prehomogeneous. Let $\Delta_0, \Delta_1, \dots, \Delta_r$ be the set of fundamental relative invariants of (B, V) , where Δ_0 is the unique fundamental relative invariant of (G, V) (the unicity is due to the one-dimensional quotient hypothesis we make here). We denote by d_i (resp. λ_i) the degree (resp. the infinitesimal character) of Δ_i ($i = 0, \dots, r$). Let \mathfrak{b} be the Lie algebra of B . Let $\mathfrak{t} \subset \mathfrak{b}$ be a Cartan subalgebra of \mathfrak{g} , and let Σ be the set of roots of $(\mathfrak{g}, \mathfrak{t})$. Denote by W the Weyl group of Σ . Let Σ^+ be the set of positive roots such that $\mathfrak{b} = \mathfrak{t} \oplus \sum_{\alpha \in \Sigma^+} \mathfrak{g}^\alpha$. Let $\mathfrak{a}^* = \bigoplus_{i=0}^r \mathbb{C}\lambda_i \subset \mathfrak{t}^*$ and let $\rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} \alpha$. Define $A = \mathfrak{a}^* + \rho \subset \mathfrak{t}^*$. F. Knop [4,1] has proved that there exists an isomorphism (the so-called Harish Chandra isomorphism) $h : \mathcal{T}_0 = \mathbf{D}(V)^G \rightarrow \mathbb{C}[A]^{W_0}$, where W_0 is a subgroup of the stabilizer of A in W . If $\lambda = \sum_{i=0}^r \alpha_i \lambda_i \in \mathfrak{a}^*$, then $h(E)(\lambda + \rho) = \sum_{i=0}^r \alpha_i d_i$. Let $M = \{\mu \in A \mid h(E)(\mu) = 0\}$ and let $I(M) = \{P \in \mathbb{C}[A]^{W_0} \mid P|_M = 0\}$. We still denote by τ the automorphism of $\mathbb{C}[A]^{W_0}$ which corresponds to $\tau|_{\mathcal{T}_0}$ via the Harish Chandra isomorphism h . Let $\mathbb{C}[A]^{W_0, \tau}$ be the set of τ -invariants in $\mathbb{C}[A]^{W_0}$. We prove in fact that $\mathbb{C}[A]^{W_0} = \mathbb{C}[A]^{W_0, \tau} \oplus I(M)$ and the pullback by h of this decomposition proves the first assertion. The second assertion is a direct consequence of the first. \square

Definition 2.7. Let \mathbf{A} be a commutative associative algebra over \mathbb{C} , with unit element 1 and without zero divisors. Let $f, u \in \mathbf{A}[t]$ be two polynomials in one variable with coefficients in \mathbf{A} . Let $n \in \mathbb{N}^*$.

- 1) (Smith [9] (over \mathbb{C})). The (generalized) Smith algebra $S(\mathbf{A}, f, n)$ is the associative algebra over \mathbf{A} with generators (x, y, e) subject to the relations $[e, x] = nx$, $[e, y] = -ny$, $[y, x] = f(e)$.
- 2) The algebra $U(\mathbf{A}, u, n)$ is the associative algebra over \mathbf{A} with generators $\tilde{x}, \tilde{y}, \tilde{e}$ subject to the relations $[\tilde{e}, \tilde{x}] = n\tilde{x}$, $[\tilde{e}, \tilde{y}] = -n\tilde{y}$, $\tilde{x}\tilde{y} = u(\tilde{e})$, $\tilde{y}\tilde{x} = u(\tilde{e} + n)$.

One can show that if $f \in \mathbf{A}[t]$ is given, and if $u \in \mathbf{A}[t]$ is the unique polynomial (up to scalars in \mathbf{A}) given by $f(t) = u(t + n) - u(t)$, then $U(\mathbf{A}, u, n) \simeq S(\mathbf{A}, f, n)/(\Omega)$, where $\Omega = xy - u(e)$ generates the center of $S(\mathbf{A}, f, n)$ over \mathbf{A} .

As $XY \in \mathcal{T}_0$, we know from Theorem 2.6 that there exists a unique $u_{XY} \in \mathcal{Z}(\mathcal{T})[t]$ such that $XY = u_{XY}(E)$. Our main result is then the following:

Theorem 2.8. *The mapping $\tilde{x} \mapsto X, \tilde{y} \mapsto Y, \tilde{e} \mapsto E$ extends uniquely to an isomorphism of $\mathcal{Z}(\mathcal{T})$ -algebras between $U(\mathcal{Z}(\mathcal{T}), u_{XY}, d_0)$ and $\mathcal{T}_0[X, Y] = \mathbf{D}(V)^{G'}$.*

Sketch of proof. From Theorem 2.6 we know that $\mathcal{T}_0[X, Y] = \mathbf{D}(V)^{G'}$ is generated over $\mathcal{Z}(\mathcal{T})$ by X, Y, E . As these three elements satisfy the defining relations of $U(\mathcal{Z}(\mathcal{T}), u_{XY}, d_0)$, there exists a surjective morphism of $\mathcal{Z}(\mathcal{T})$ -algebras $\varphi : U(\mathcal{Z}(\mathcal{T}), u_{XY}, d_0) \rightarrow \mathcal{T}_0[X, Y]$. Theorem 2.6 is used again to prove that any $D \in \mathcal{T}_0[X, Y]$ can be written uniquely in the form $D = \sum_{\ell > 0, k \geq 0} \alpha_{k, \ell} Y^\ell E^k + \sum_{m \geq 0, r \geq 0} \beta_{m, r} X^m E^r$ with $\alpha_{k, \ell}, \beta_{m, r} \in \mathcal{Z}(\mathcal{T})$. This implies that φ is bijective.

We define the radial component \bar{D} of an operator $D \in \mathcal{T}_0[X, Y] = \mathbf{D}(V)^{G'}$, by $D(\psi \circ \Delta_0) = (\bar{D}\psi) \circ \Delta_0$, for $\psi \in \mathbb{C}[t]$. If $D \in \mathcal{Z}(\mathcal{T})$, then \bar{D} is just a constant. If $u(t) = D_0 + D_1 t + \dots + D_k t^k \in \mathcal{Z}(\mathcal{T})[t]$, we define $\bar{u} \in \mathbb{C}[t]$, by $\bar{u}(t) = \bar{D}_0 + \bar{D}_1 t + \dots + \bar{D}_k t^k$. Let us also denote by R the image of the map $D \mapsto \bar{D}$ (the so-called algebra of radial components). As a consequence we obtain a new proof of the following result. \square

Corollary 2.9. (See Levasseur [6].) $R \simeq U(\mathbb{C}, \bar{u}_{XY}, d_0)$.

3. The case of irreducible regular PV 's of parabolic commutative case

Let $\tilde{\mathfrak{g}} = V^- \oplus \mathfrak{g} \oplus V^+$ be a 3-grading of a simple Lie algebra $\tilde{\mathfrak{g}}$ over \mathbb{C} ($[\mathfrak{g}, V^\pm] \subset V^\pm, [V^-, V^+] \subset \mathfrak{g}$). Then if G is the connected subgroup of the adjoint group of $\tilde{\mathfrak{g}}$ with Lie algebra \mathfrak{g} , it is well known that (G, V^+) is a multiplicity free space (these PV 's are said to be of *commutative parabolic type*, they are in one to one correspondence with Jordan triple systems or with hermitian symmetric spaces). We assume further that (G, V^+) is regular as a PV (these objects are in one to one correspondence with simple Jordan algebras, or with hermitian symmetric spaces of tube type). These regular PV 's of commutative parabolic type form a subclass of the multiplicity free spaces with one dimensional quotient. For these we can give some further information concerning the preceding algebras of differential operators. First of all, the number of Δ_i 's is equal to d_0 . In the notations of Section 2, $\Delta_0, \Delta_1, \dots, \Delta_{d_0-1}$ is the set of fundamental relative invariants of (B, V^+) . For simplicity define $n = d_0 - 1$. For $\mathbf{a} = (a_0, \dots, a_n) \in \mathbb{N}^{n+1}$, let $V_{\mathbf{a}}$ be the irreducible G -submodule of $\mathbb{C}[V^+]$ generated by $\Delta^{\mathbf{a}} = \Delta_0^{a_0} \dots \Delta_n^{a_n}$. It is well known that the G -module $V_{\mathbf{a}} \otimes V_{\mathbf{a}}^*$ contains up to constants, a unique G -invariant vector $C_{\mathbf{a}}$, which can be viewed as a G -invariant differential operator (see [2]). These operators are sometimes called Capelli operators. Define $\mathbf{1}_j = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{N}^{n+1}$. Note that $XY = C_{\mathbf{1}_0}$ and that $E = C_{\mathbf{1}_n}$.

Proposition 3.1.

- 1) Let $\mathbb{C}[E, X, X^{-1}, Y]$ be the associative subalgebra of $\mathbf{D}(V^+)$ generated by E, X, X^{-1}, Y . We have $\mathbb{C}[E, X, X^{-1}, Y] = \mathcal{T}$.
- 2) (Yan [10].) We have $\mathcal{T}_0 = \mathbb{C}[XY, C_{\mathbf{1}_1}, \dots, C_{\mathbf{1}_n}]$, and $XY, C_{\mathbf{1}_1}, \dots, C_{\mathbf{1}_n}$ are algebraically independent.
- 3) We have $\mathcal{T}_0[X, Y] = \mathbf{D}(V^+)^{G'} = \mathbb{C}[X, Y, C_{\mathbf{1}_1}, \dots, C_{\mathbf{1}_n}]$.

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